Solution of series of Exercises 1

Exercise 1:

a). - Positivity: $d(x,y) = \|x-y\| \geq 0$, moreover, $d(x,y)=0 \Leftrightarrow \|x-y\|=0 \Leftrightarrow x=y$.

- Symmetry: $\forall x,y \in E$, we have $d(x,y) = \|x-y\| = \|y-x\||-1| = d(y,x)$.

- triangular inequality: $\forall x,y,z \in E$ we have: $d(x,y) = \|x-y\| \leq \|x-z\| + \|z-y\| \leq d(x,z) + d(z,y)$.

b). Let d be a metric on E, we claim that $\|\cdot\|_d = d(\cdot,0) : E \times E \to \mathbb{R}^+$ is a norm on E such that $d(x,y) = \|x-y\|$, $\forall x,y \in E$. We show that $\|\cdot\|_d$ is a norm.

- The positivity of $\|\cdot\|_d$ is clear.

- Homogeneity: Let $\lambda \in \mathbb{K}$, $x \in E$: then $\|\lambda x\|_d = d(\lambda x, 0) = |\lambda| d(x,0) = |\lambda| \|x\|_d$.

- triangular inequality: Let $x,y \in X$, then

$\|x+y\| = d(x+y,0) \leq d(x+y, x+y) + d(y,0) = d(x,0) + d(y,0) = \|x\|_d + \|y\|_d$.

So $\|\cdot\|_d$ is a norm on E. Moreover, for every $x,y \in E$, we have:

$\|x-y\|_d = d(x-y,0) = d(x+(-y), y+(-y)) = d(x,y)$ (our claim follows).

c). Let $\|\cdot\|$ be a norm on E. Consider $((\lambda_n, x_n))_n \subset \mathbb{K} \times E$ such that $(\lambda_n, x_n) \longrightarrow (\lambda, x)$. Then

$\|\lambda x - \lambda_n x_n\| \leq \|\lambda x - \lambda_n x\| + \|\lambda_n x - \lambda_n x_n\| = |\lambda - \lambda_n| \|x\| + |\lambda_n| \|x - x_n\| \longrightarrow 0 \ (n \to \infty)$.

- Let $(x_n, y_n) \longrightarrow (x,y) \in E \times E$ then:

$\|x_n + y_n - (x+y)\| \leq \|x_n - x\| + \|y_n - y\| \longrightarrow 0 \ (n \to \infty)$.

d). Let $\|\cdot\|_1$ and $\|\cdot\|_2$ two equivalent norms on E inducing respectively the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on E. Let $U \in \mathcal{T}_1$, we show that $U \in \mathcal{T}_2$. Indeed, fix any $x \in U$, there exists $r_1 > 0$ such that $B_1(x, r_1) \subset U$. Since $\|\cdot\|_2$ is equivalent to $\|\cdot\|_1$, there exists $C > 0$ such that $C\|z\|_1 \leq \|z\|_2$ $\forall z \in E$. Hence

$B_2(x, r_2) \subset B_1(x, r_1)$: $r_2 = C r_1$. Then $U \in \mathcal{T}_2$, i.e. $\mathcal{T}_1 \subset \mathcal{T}_2$.

Analogously, we show that $\forall V \in \mathcal{T}_2 \Rightarrow V \in \mathcal{T}_1$ i.e. $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Exercise 2:

1. i. Let $f \in E$: $\|f\|_1 = \sup(|f(x)| + |f'(x)|) = 0$ iff $|f(x)| = 0$ and $|f'(x)| = 0$ $\forall x \in [0,1]$.

iff $f \equiv 0$

ii. $\|\lambda f\|_1 = \sup(|\lambda f(x)| + |\lambda f'(x)|) = |\lambda| \sup(|f(x)| + |f'(x)|) = |\lambda| \|f\|_1$

iii. $\forall f,g \in E$, we have: $\|f+g\|_1 = \sup(|f(x)+g(x)| + |f'(x)+g'(x)|) \leq \sup(|f(x)| + |f'(x)|) + \sup(|g(x)| + |g'(x)|)$

then $\|f+g\|_1 \leq \|g\|_1 + \|f\|_1$.

Thence $\|\cdot\|_1$ is a norm on E.

2) In one hand we have: $|f(x)| + |f'(x)| \leq \sup|f(x)| + \sup|f'(x)| \Rightarrow \|f\|_1 \leq \|f\|_2$ --- ①

- In other hand: $|f(x)| \leq |f(x)| + |f'(x)| - |f'(x)| \leq \sup(|f(x)| + |f'(x)|) - |f'(x)|$

this $\sup|f(x)| \leq \sup(|f(x)| + |f'(x)|) - |f'(x)|$

implies $|f'(x)| \leq \sup|f|$

which implies that: $|f'(x)| \le \sup_I p(|f(x)|+|f'(x)|) - \sup_I p|f(x)|$ $I=[0,1]$.

then we have: $\sup p|f'(x)| \le \sup p(|f(x)|+|f'(x)|) - \sup p|f(x)|$

Hence: $\sup p|f(x)| + \sup p|f'(x)| \le \sup p(|f(x)|+|f'(x)|)$ i.e $\|f\|_2 \le \|f\|_1$ ... ②

From ① and ② it follows: $\|f\|_1 = \|f\|_2$.

Exercise 3.

1- Show that N is a norm on E.

i)- $N(f)=0 \iff \int_0^1 t|f(t)|dt = 0 \iff t|f(t)|=0 \iff |f(t)|=0\ \forall t\neq 0$ or $t=0$.

if we suppose that $f(0)\neq 0$ we obtain a contradiction because f is continuous on $[0,1]$. (o).

ii) $N(f+g) = \int_0^1 t|f+g(t)|dt \le \int_0^1 t|f(t)|dt + \int_0^1 t|g(t)|dt \le N(f)+N(g)$.

iii) $N(\lambda f) = \int_0^1 t|\lambda f(t)|dt = |\lambda|\int_0^1 t|f(t)|dt = |\lambda| N(f)$.

2- a) $\|f_n\|_1 = \sup_{[0,1]} |(1-nt)| = 1$. N

$N(f_n) = \int_0^{1/n} t(1-nt)\,dt = \left(\frac{1}{2n^2} - \frac{n}{3n^3}\right) = \frac{1}{6n^2}$ ⇒

b) Since $\lim \left(\frac{\|f_n\|_1}{N(f_n)} = \frac{1}{1/6n^2}\right) = \lim 6n^2 = \infty$, then N and $\|\cdot\|_1$ are not equivalent.

Exercise 4

Let consider the function sequence $f_n(t) = t^n\sqrt{n}$

$\|f\|_\infty = \sqrt{n}$, $\|f\|_1 = \int_0^1 t^n\sqrt{n} = \frac{\sqrt{n}}{n+1}$, $\|f\|_2 = \int_0^1 n\, t^{2n} = \frac{n}{2n+1}$. Then

$\lim\left(\frac{\|f\|_1}{\|f\|_\infty} = \frac{\sqrt{n}}{(n+1)\sqrt{n}}\right)=0$, $\lim\left(\frac{\|f\|_2}{\|f\|_\infty} = \frac{\sqrt{n}}{2n+1}\right)=0$, $\lim\left(\frac{\|f\|_1}{\|f\|_2} = \frac{\sqrt{n}\,(2n+1)}{(n+1)n}\right)=0$

then the three norms are not equivalent.

Exercise 6

1- Let consider $\varphi(x) = \frac{rx}{1+\|x\|}$.

• Firstly we have $\varphi(E) \subset B(0,r)$.

• Secondly, φ is bijective mapping; because:

a)- $\forall x,y \in E$, if $\varphi(x)=\varphi(y)$, then $\frac{x}{1+\|x\|} = \frac{y}{1+\|y\|} \iff x = \frac{1+\|x\|}{1+\|y\|}\, y = \lambda y$;

then $|\lambda|\|y\| = \|y\| \Rightarrow |\lambda|=1$; $\|x\|=\|y\|$, and $\|x-y\| = \left(\frac{1+\|x\|}{1+\|y\|}-1\right)\|y\|$

Hence $x=y$ ($\|\cdot\|$ is a norm). i.e φ is injective.

b)- $\forall y \in B(0,r)$, let consider $x_0 = \alpha y$; so $\varphi(x_0) = \alpha\varphi(y) = \frac{\alpha r y}{1+\alpha\|y\|}$, $\alpha>0$

then, $\varphi(x_0) = \frac{ry}{\frac{1}{\alpha}+\|y\|}$; if we take $\frac{1}{\alpha} = r-\|y\|$ we get $\varphi(x_0)=y$ then φ is surjective. ($\|x_0\| = \frac{\|y\|}{r-\|y\|}$).

- a)+b) ⇒ φ is bijective and $\varphi^{-1}(y) = \frac{y}{r-\|y\|}$

d is continuous because [illegible] and [illegible] are continuous.

d' is continuous because [illegible] and [illegible] [illegible]

So d is a homeomorphism.

Exercise 6:

1. $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and $\|f\|_1 = \|f'\|_\infty + |f(0)|$ are norms on E.

$\|f\| = 0$ iff $\|f\|_\infty = 0$ and $\|f'\|_\infty = 0$ iff $f = 0$.

$\|f+g\| = \|f+g\|_\infty + \|f'+g'\|_\infty \le \|f\|_\infty + \|f'\|_\infty + \|g\|_\infty + \|g'\|_\infty = \|f\| + \|g\|$.

$\|\lambda f\| = |\lambda|(\|f\|_\infty + \|f'\|_\infty) = |\lambda| \|f\|$.

Then $\|\cdot\|$ is a norm on E.

$\|f\|_1 = 0$ iff $\|f'\|_\infty = 0$ and $|f(0)| = 0$ iff f is constant and $f(0) = 0$ iff $f = 0$.

$\|f+g\|_1 = \|f'+g'\|_\infty + |f(0)+g(0)| \le \|f'\|_\infty + |f(0)| + \|g'\|_\infty + |g(0)| = \|f\|_1 + \|g\|_1$.

$\|\lambda f\|_1 = |\lambda| \|f'\|_\infty + |\lambda| |f(0)| = |\lambda|(\|f'\|_\infty + |f(0)|) = |\lambda| \|f\|_1$.

Then $\|\cdot\|_1$ is a norm on E.

2. $\|\cdot\|$ is equivalent to $\|\cdot\|_1$.

$\forall f \in E$ we have $f(x) = f(0) + \int_0^x f'(t)\,dt$, then

$\|f\|_\infty \le |f(0)| + \|f'\|_\infty$, then $\|f\|_\infty + \|f'\|_\infty \le 2(|f(0)| + \|f'\|_\infty)$ so $\|f\| \le 2\|f\|_1$.

Because $|f(0)| \le \|f\|_\infty$, then $\|f\|_1 = \|f'\|_\infty + |f(0)| \le \|f\|_\infty + \|f'\|_\infty = \|f\|$ ($\|f\|_1 \le \|f\|$).

Hence the two norms are equivalent.

3. a) Let be (f_n) is a Cauchy sequence in $(E, \|\cdot\|_1)$ i.e. $\forall \varepsilon > 0, \exists N(\varepsilon), \forall n, m \ge N$, $\|f_n - f_m\|_1 \le \varepsilon$

Then we have $\forall \varepsilon > 0, \exists N(\varepsilon), \forall n > m \ge N$, $\|f'_n - f'_m\|_\infty + |f_n(0) - f_m(0)| \le \varepsilon$

then $\|f'_n - f'_m\|_\infty \le \varepsilon$ and $|f_n(0) - f_m(0)| \le \varepsilon$, then (f'_n) is a Cauchy sequence in $(F, \|\cdot\|_\infty)$

that is complete, so f'_n is convergent in F towards limit noting g

i.e. $\lim \|f'_n - g\|_\infty = 0$. And $(f_n(0))$ is a Cauchy sequence in $\mathbb{R}$ which is complete

so $f_n(0)$ is convergent to a limit l.

Remarking that f_n converges to f in $(E, \|\cdot\|)$ where $f(x) = l + \int_0^x g(t)\,dt$

Then $f' = g$ and $l = f(0)$, moreover we have:

$\|f_n - f\|_1 = \|f'_n - f'\|_\infty + |f_n(0) - f(0)| = \|f'_n - g\|_\infty + |f_n(0) - l| \to 0$ $(n \to \infty)$.

So, f_n converges towards f in $(E, \|\cdot\|_1)$, Hence $(E, \|\cdot\|_1)$ is Banach space.

b) Because $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent norms so $(E, \|\cdot\|)$ is a Banach space.

Exercise 7:

1. Firstly we remark that (u_n) are continuous ($u_n \in E$). It is a Cauchy sequence. Indeed

for n, let $p > q$, we estimate $\|u_p - u_q\|_1$. We have:

$u_p(x) - u_q(x) = 0$ on $[-1, -\frac{1}{q}] \cup [0, 1]$, $u_p(x) - u_q(x) = -(qx+1)$ on $]-\frac{1}{q}, -\frac{1}{p}[$

and $u_p(x) - u_q(x) = (p-q)x$ on $[-\frac{1}{p}, 0[$.

where : $\|u_p - u_q\|_2^2 = \int_{-1}^{1} |u_p - u_q|^2 \, dx$

$$= \int_{-\frac{1}{q}}^{-\frac{1}{p}} |qx+1|^2 \, dx + (p-q)^2 \int_{-\frac{1}{p}}^{0} x^2 \, dx = \frac{1}{q}\int_{0}^{\frac{p-q}{p}} t^2 \, dt + (p-q)^2 \int_{-\frac{1}{p}}^{0} t^2 \, dt$$

$$= \frac{1}{3qp^3}(p-q)^3 + \frac{(p-q)^2}{3p^3} \leq \frac{1}{3q} + \frac{1}{3p} = \frac{2}{3q} \xrightarrow[q \to \infty]{} 0 . \quad ((u_n) \text{ is a Cauchy seq})$$

2°- It is clear that (u_n) converges point wise to u such that :

$$u(x) = \begin{cases} 0 \ ; \text{ if } -1 \leq x < 0 \\ 1 \ , \text{ if } 0 \leq x < 1 . \end{cases}$$

We have $\|u - u_n\|_2^2 = \int_{-1/n}^{0} (nx+1)^2 \, dx = \frac{1}{n}\int_{0}^{1} t^2 \, dt = \frac{1}{3n} \longrightarrow 0 \ (n \to +\infty)$.

Finally, we remark that u is not continuous, then $u \notin E$.

3°- $(E, \|\cdot\|)$ is not complete, then it is not a Banach space.

Exercise 8:

1°- $f \in C^0(\mathbb{R};\mathbb{R})$ and $\lim_{|x| \to \infty} f(x) = 0$ i.e $\forall \varepsilon > 0, \exists A >> 0 : \forall x > A \Rightarrow |f(x)| < \varepsilon$. Because f is a continuous function, then it uniformelly continuous on $[A, A]$; so there exists $x_0 \in [A, A]$: $\sup_{[A,A]} |f(x)| = |f(x_0)|$. Then $\forall x \in \mathbb{R}$ $|f(x)| \leq \max\{|f(x_0)|, \varepsilon\}$. (It is a bounded).

2°- It is clear that $\|\cdot\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$ is a norm, because $\forall \varepsilon > 0$, $\exists A >> 0 : \sup_{[-A,A]} |f(x)| = 0$ and $\forall |x| > A \Rightarrow |f(x)| = 0 \Rightarrow \forall x \in [-A, A]$ $f(x) = 0$ then $f(x) = 0$ $\forall x \in \mathbb{R}$.

3°- a). $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall p > q \geq n_0 \Rightarrow \|f_p - f_q\| := \sup_{\mathbb{R}} |f_p(x) - f_q(x)| < \varepsilon$,

So, $|f_p(x) - f_q(x)| < \varepsilon$ that is a Cauchy sequence in $\mathbb{R}$, that is a complete, then there exist

b) Then there exists $f : \mathbb{R} \to \mathbb{R}$: $f(x) = \lim_{n \to \infty} f_n(x)$ such that f_n converges point wise to f.

c) Because $\lim_{|x| \to \infty} f_n(x) = 0$ then $\lim_{|x| \to \infty} f(x) = 0$ ($|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)|$).

· $f_n \to f$ point wise thi $\forall \varepsilon > 0$, $\exists A >> 0$: $\forall x \in |x| > A \Rightarrow |f_n(x) - f(x)| < \varepsilon$

And $|f(x) - f_n(x)| \xrightarrow[n \to \infty]{} 0$ on $[-A, A]$, or $\exists x_0 \in [-A, A]$: $\sup_{I} |f(x)| = |f(x_0)|$

then $\|f(x) - f_n(x)\|_\infty = |f_n(x_0) - f(x_0)| \xrightarrow[n \to \infty]{} 0$. $\Rightarrow f_n \longrightarrow f$ uniformelly on $\mathbb{R}$.

d). let be $x_0 \in \mathbb{R}$. We have:

$$|f(x) - f(x_0)| \leq \underbrace{|f(x) - f_n(x)|}_{①} + \underbrace{|f_n(x) - f_n(x_0)|}_{②} + \underbrace{|f_n(x_0) - f(x_0)|}_{③}$$

① $< \varepsilon/3$ because $f_n(x) \to f(x)$ $\forall x \in \mathbb{R}$.

② $< \varepsilon/3$ f_n is continuous at x_0.

③ $< \varepsilon/3$ because $f_n(x_0) \longrightarrow f(x_0)$

$\Rightarrow f$ is continuous function on $\mathbb{R}$.

(e). We conclude that $C^0(\mathbb{R};\mathbb{R})$ is a Banach space.