CHAPTER 1

BANACH SPACES

In this chapter, we will give some generalities about abstract normed spaces, with examples, and we will treat the case of finite-dimensional spaces. Some topological notions are added in order to simplify comprehension and resolution of exercises in fifth and sixth semesters. This will also be an opportunity to set certain notations.

1.1 Normed vector spaces

1.1.1 Norm

Definition 1.1.1. *Let E be a real or complex vector space. A norm on E is an application, most often denoted $\|.\|$:*

$$\|\cdot\| : E \longrightarrow \mathbb{R}_+ = [0, +\infty[$$

having the following three properties:

1. *a)* $\|x\| \geq 0$ *for all* $x \in E$ *and* *b)* $\boxed{\|x\| = 0 \Longleftrightarrow x = 0}$;
2. $\boxed{\|\lambda x\| = |\lambda| \|x\|}$, $\forall x \in E, \forall \lambda \in \mathbb{K}$ *(homogeneity);*
3. $\|x + y\| \leqslant \|x\| + \|y\|, \forall x, y \in E$ *(triangular inequality).*

If we delete 1) b), we say that $\|.\|$ is a semi-norm. Note that then 2) nevertheless results in $\|0\| = 0$.

1.1.2 Norm proprieties

Proposition 1.1.1. *The function $x \in E \mapsto \|x\| \in \mathbb{R}_+$ is continuous*

Proof. Just use inequality $|\,\|x\| - \|y\|\,| \leq \|x - y\|$. □

From a norm, we obtain a distance on E by setting $\boxed{d(x, y) = \|x - y\|}$.

We then define the :

- Open balls : $\overset{\circ}{B}(x, r) = \{y \in E; \|x - y\| < r\}$;
- Closed balls : $B(x, r) = \{y \in E; \|x - y\| \leqslant r\}$,

which makes it possible to define a topology on E; a part A of E is open (and we also say that A is an open set of E) if for all $x \in A$ there exists a ball centered at x, of radius $r = r_x > 0$, contained in A. There is no need to specify whether it is an open ball or a closed ball. Indeed, if A contains the closed ball $B(x, r)$, it contains a fortiori the open ball $\overset{\circ}{B}(x, r)$; and, conversely, if A contains the open ball $\overset{\circ}{B}(x, r)$, it contains the closed ball $B(x, r')$, for all $r' < r$. Note that the empty set $\varnothing$ is an open set (since there is no x in A, the property defining open set is trivially verified). The entire space E is clearly an open space. It follows from the definition that any union of open parts is an open set. Any intersection of a finite number of open sets is an open one. If $x \in A = A_1 \cap \cdots \cap A_n$, and $\overset{0}{B}(x, r_k) \subseteq A_k$, then $\overset{\circ}{B}(x, r) \subseteq A$, with $r = \min(r_1, \ldots, r_n)$

A part V containing the point $x_0 \in E$ is a neighborhood of x_0 if it contains a ball (open or closed) with center x_0, and with radius $r > 0$.

A part is closed (we also say that it is closed set) if its complement is open. By complementarity, we obtain that $\varnothing$ and E are closed, that the intersection of any family of closed sets is still closed one, as well as any union of a finite number of closed sets. If $A \subseteq E$ is a part of E, we call interior of A, and we write $\overset{\circ}{A}$, or $int(A)$, the largest open set contained in

A (it is the union of all the open sets contained in A), and we call closure, of A the smallest closed set containing A (it is the intersection of all closed sets containing A). We denote by $\bar{A}$ the closure of A. We recall (it's easy to see) that $x \in \bar{A}$ if, and only if, there exists a sequence of elements of A converging to x. We say that A is dense in E if $\bar{A} = E$.

Proposition 1.1.2. *Any open ball is an open set and any closed ball is a closed set.*

Proof. 1) Let $x \in \overset{\circ}{B}(x_0, r_0)$ and let $0 < r < r_0 - \|x - x_0\| > 0$. For $\|x - y\| \leqslant r$, we have $\|y - x_0\| \leq \|y - x\| + \|x - x_0\| \leq r + \|x - x_0\| < r_0$; therefore $B(x, r) \subseteq \overset{\circ}{B}(x_0, r_0)$.
2) Let $x \notin B(x_0, r_0)$ and let $0 < r < \|x - x_0\| - r_0$. Since, if $\|y - x\| \leq r$, we have $\|y - x_0\| \geq \|x_0 - x\| - \|x - y\| \geq \|x_0 - x\| - r > r_0$, then $B(x, r) \subseteq [B(x_0, r_0)]^c$. $\square$

Remarks.

(i) Closed subsets are important while studying the solution of equation, where one looks for approximate solutions by constructing sequences of approximations, of all which belong to a set Y of functions with certain properties. If Y is a closed set and if the sequence is convergent, the limit also belongs to Y, giving a convergent sequence of approximations in the solution set Y.
(ii) It is clear that $Y \subset \bar{Y}$, and $Y = \bar{Y}$ if and only if Y is closed.
All the preceding topological notions do not involve the fact that E is a vector space, nor that the distance is defined from a norm; they are therefore valid in any metric space. On the other hand, we have a specific property in normed spaces, which justifies the notation of open balls: the interior of $B(r, r)$ is the open ball $\overset{\circ}{B}(x, r)$ and the closure of the open ball $\overset{\circ}{B}(x, r)$ is the closed ball $\overline{B(x, r)}$ (see below).

Definition 1.1.2. *When a vector space E is endowed with a norm and the topology associated with this norm, we say that it is a normed vector space, or, more simply, a normed space.*

Notation. We will denote by B_E the closed ball $B(0, 1)$ with center 0 and radius 1. We will say that it is the **unit ball** of E.

Proposition 1.1.3. *If E is a normed space, then the mapping:*

$$+: \quad E \times E \to E \qquad \text{and} \qquad \mathbb{K} \times E \to E$$
$$(x, y) \mapsto x + y \qquad\qquad (\lambda, x) \mapsto \lambda x$$

are continuous.

Definition 1.1.3. *Let E be a real or complex vector space, provided with a topology. We say that E is a topological vector space (t.v.s) if the maps:*

$$+: \quad E \times E \to E \qquad \text{and} \qquad \mathbb{K} \times E \to E$$
$$(x, y) \mapsto x + y \qquad\qquad (\lambda, x) \mapsto \lambda x$$

are continuous.

We say that a topological vector space is locally convex space (l.c.s), if every point has a base of convex neighborhoods.

Balls are convex, and any normed space is a (t.v.s) locally convex.

Corollary 1.1.1. *The translations:*

$$\tau_a: \quad E \longrightarrow E \qquad (a \in E)$$
$$x \longmapsto x + a$$

and the dilations:

$$h_\lambda: \quad E \longrightarrow E \qquad (\lambda \in \mathbb{K})$$
$$x \longmapsto \lambda x$$

are continuous. These are homeomorphisms (if $\lambda \neq 0$ for dilations).

Corollary 1.1.2. *All closed balls of radius $r > 0$ are homeomorphic to each other, therefore to B_E. All open balls of radius $r > 0$ are homeomorphic to each other.*

Corollary 1.1.3. *The closure of the open ball* $\overset{\circ}{B}(x,r)$ *is the closed ball* $B(x,r)$ *and the interior of the closed ball* $B(x,r)$ *is the open ball* $\overset{\circ}{B}(x,r)$ *.*

Proof. 1) The closure of the open ball is obviously contained in the closed ball, since the latter is closed in E. Conversely, if $y \in B(x,r)$, we have
$y_n = \frac{1}{n}x + \left(1-\frac{1}{n}\right)y \in \overset{\circ}{B}(x,r)$ car $\left\|x - \left[\frac{1}{n}x + \left(1-\frac{1}{n}\right)y\right]\right\| = \left(1-\frac{1}{n}\right)\|x-y\| < r$; as $y = \lim_{n\to\infty} y_n$, we obtain $y \in \overset{\circ}{B}(x,r)$.

2) Being open in E, the open ball is contained in the interior of the closed ball. To show the reverse inclusion, show that if y is not in the open ball, then no ball $B(y,\rho)$ of center y and radius $\rho > 0$ is contained in $B(x,r)$. But if y is not in $\overset{\circ}{B}(x,r)$, then we have $\|y-x\| \geq r$. For all $\rho > 0$, the vector $z = y + \frac{\rho}{\|y-x\|}(y-x)$ is in $B(y,\rho)$, since $\|z-y\| = \frac{\rho}{\|y-x\|}\|y-x\| = \rho$, but is not in $B(x,r)$, because $\|z-x\| = \left\|y + \frac{\rho}{\|y-x\|}(y-x) - x\right| = \left(1+\frac{\rho}{\|y-x\|}\right)\|y-x\| \geq \left(1+\frac{\rho}{|y-x|}\right)r > r$. So y is not in the interior of $B(x,r)$. □

Corollary 1.1.4. *If* F *is a vector subspace of* E*, then its closure* $\bar{F}$ *is a vector subspace too.*

Proof. Let be $x, y \in \bar{F}$ and $a, b \in \mathbb{K}$. There exist $x_n, y_n \in F$ such that $x_n \underset{n\to\infty}{\longrightarrow} x$ and $y_n \underset{n\to\infty}{\longrightarrow} y$. By Proposition 1.4.1, we have $ax+by = \lim_{n\to\infty}(ax_n + by_n)$; and as $ax_n + by_n \in F$, we obtain $ax + by \in \bar{F}$. □

1.1.3 Some common examples

Spaces of sequences

1) a) It is immediate to see that if we put, for $x = (x_1, \ldots, n) \in \mathbb{K}^n$:

$$\begin{cases} \|x\|_1 = |x_1| + \cdots + |x_n| \\ \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\} \end{cases}$$

Then $\|.\|_1$ and $\|.\|_\infty$ are two norms on $\mathbb{K}^n$.

We note $\ell_1^n = (K^n, |.|_1)$ and $\ell_\infty^n = (K^n, |.|_\infty)$.

b) If p is an real number that satifies $1 < p < \infty$, a norm on $\mathbb{R}^n$ is obtained when we put:

$$\boxed{\|x\|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}}$$

We note $\ell_p^n = (K^n, |.|_p)$. Only triangular inequality:

$$\boxed{\left(\sum_{k=1}^n |x_k + y_k|^p\right)^{1/p} \leqslant \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^n |y_k|^p\right)^{1/p},}$$

called **Minkowski** inequality, is not obvious; it can be demonstrated as follows: By convexity of the function $t \in \mathbf{R}_+ \mapsto t^p$, we have $[\alpha u + (1-\alpha)v]^p \leqslant \alpha u^p + (1-\alpha)v^p$ if $0 \leqslant \alpha \leqslant 1$ and $u, v \geqslant 0$. Take $\alpha = \frac{\|x\|_p}{\|x\|_p + \|y\|_p}$ (such that $1 - \alpha = \frac{\|y\|_p}{\|x\|_p + \|y\|_p}$),$u = \frac{|x_k|}{\|x\|_p}$ and $v = \frac{|y_k|}{\|y\|_p}$ (if $\|x\|_p = 0$ or $\|y\|_p = 0$, the result is abvious). By summing, we get

$$\frac{1}{(\|x\|_p + \|y\|_p)^p} \sum_{k=1}^n (|x_k| + |y_k|)^p \leqslant 1,$$

which gives the result, since $|x_k + y_k| \leqslant |x_k| + |y_k|$ for all $k = 1, \ldots, n$.

A very useful inequality is the **Hölder's inequality** . Recall that if $1 < p < \infty$, **the conjugate exponent** of p is the number q satisfying $\boxed{\frac{1}{p} + \frac{1}{q} = 1}$. Explicitly, $\boxed{q = \frac{p}{p-1}}$. We have $1 < q < \infty$, and p is the conjugate exponent of q. They are also linked by the equality $\boxed{(p-1)(q-1) = 1}$. Hölder's inequality is then stated as follows if $1 < p < \infty$ and q is the conjugate exponent of p, then, for all $x_1, ..., x_n, y_1, ..., y_n \in \mathbb{K}$, we have:

$$\boxed{\sum_{k=1}^n |x_k y_k| \leqslant \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} \left(\sum_{k=1}^n |y_k|^q\right)^{1/q}}$$

If $p = 2$, then $q = 2$: this is **the Cauchy-Schwarz inequality** (due, in this form, to Cauchy in 1821).

To show Hölder's inequality, we start from the inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, for $a, b \geq 0$ (this is a consequence of the convexity of the function $t \in \mathbb{R}_+ \mapsto \frac{t^p}{p}$ and of the fact that its derivative $t \mapsto t^{P-1}$ is the reciprocal (or inverse) of the derivative $t \mapsto t^{q-1}$ of $t \mapsto \frac{t^q}{q}$, as we can see it just as simply, for example by studying the variations of the function $t \mapsto \frac{t^p}{p} + \frac{b^q}{q} - bt$); we apply it with $a = \frac{|x_k|}{\|x\|_p}$ and $b = \frac{|y_k|}{\|y\|_q}$ (we can assume $\|x\|_p > 0$ and $\|y\|_q > 0$), and we add up. We obtain $\frac{1}{\|x\|_p\|y\|_q} \sum_{k=1}^n |x_k y_k| \leq \frac{1}{p} + \frac{1}{q} = 1$, hence Hölder's inequality.

2) These examples generalize to infinite dimension.

a)Let:

$$\boxed{c_0 = \left\{x = (x_n)_{n\geqslant 1} \in \mathbb{K}^{N^*}; \lim_{n\to+\infty} x_n = 0\right\}},$$

and:

$$\boxed{\ell_\infty = \left\{x = (x_n)_{n\geqslant 1} \in \mathbb{K}^{\mathbf{N}^*}; (x_n)_n \text{ is bounded } \right\}};$$

we provide them with the norm defined by:

$$\boxed{\|x\|_\infty = \sup_{n\geq 1} |x_n|}.$$

b) for $1 \leq p < \infty$, we set:

$$\boxed{\ell_p = \left\{x = (x_n)_{n\geqslant 1} \in \mathbb{K}^{N^*}; \sum_{n=1}^{\infty} |x_n|^p < +\infty\right\}};$$

it is provided with the norm defined by:

$$\boxed{\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}}.$$

The fact that ℓ_p is a vector subspace of the space of sequences, and that $\|.\|_p$, i.e. a norm on ℓ_p is deduced from the Minkowski inequality (obvious when $p = 1$) generalized as follows:

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/p} \leqslant \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p},$$

For all $x_1, x_2, \ldots, y_1, y_2, \ldots \in \mathbf{K}$. We obtain it from the previous one by making the number of terms tend towards infinity: for all $N \geqslant 1$, we have: $\left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{1/p} \leqslant$

$\left(\sum_{n=1}^{N}|x_n|^p\right)^{1/p}+\left(\sum_{n=1}^{N}|y_n|^p\right)^{1/p}\leqslant(\sum_{n=1}^{\infty}|x_n|^p)^{1/p}+(\sum_{n=1}^{\infty}|y_n|^p)^{1/p}$

Hölder's inequality is generalized in the same way. If $1<p<\infty$ and if q is the conjugate exponent of p, we have:

$$\sum_{n=1}^{\infty}|x_ny_n|\leqslant\left(\sum_{n=1}^{\infty}|x_n|^p\right)^{1/p}\left(\sum_{n=1}^{\infty}|y_n|^q\right)^{1/q}.$$

In particular, when $x=(x_n)_n\in\ell_p$ and $y=(y_n)_n\in\ell_q$, we have $xy\in\ell_1$ and $\|xy\|_1\leq\|x\|_p\|y\|_q$

.

The spaces ℓ_p are in fact special cases of the Lebesgue spaces $L^p(m)$, whose definition we will recall below, corresponding to the counting measure on N^* .

Function spaces

1) a) Let A be a set and let the space $\mathscr{F}_b(A)$ be the space (which we also note $\ell_\infty(A)$ If we want to focus on the 'family of elements' aspect) of functions bounded on A, with values in $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. If we set:

$$\boxed{\|f\|_\infty=\sup_{x\in A}|f(x)|},$$

Then we have a norm, called the uniform norm. The topology associated with this norm is the topology of uniform convergence; indeed, it is clear that $\|f_n-f\|_\infty\underset{n\to\infty}{\longrightarrow}0$ if and only if $(f_n)_n$ converges uniformly on A to f .

b) Let K be a compact space and $\boxed{\mathscr{C}(K)}$ the space of continuous functions on K (with scalar values). Any continuous function on a compact being bounded, $\mathscr{C}(K)$ is a vector subspace of $\mathscr{F}_b(K)$. It is usually provided with the induced norm $\|f\|_\infty=\sup_{x\in k}|f(x)|$.

Note that, when $K=[0,1]$, for example, we can also provide $\mathscr{C}([0,1])$ with the norm defined by:

$$\|f\|_1=\int_0^1|f(t)|dt,$$

that verifies $|f|_1\leq|f|_\infty$.

c) On the space $\mathscr{C}([0,1])$ of functions k times continuously derivable on $[0,1]$, the norm can be set as fellown:

$$\|f\|^{(k)} = \max\left\{\|f\|_\infty, \|f'\|_\infty, \ldots, \left\|f^{(k)}\right\|_\infty\right\}$$

2) Lebesgue spaces.

Let $(S, \mathscr{F}, m)$ be a measured space; for $1 < p < \infty$, we denote $\boxed{\mathscr{L}^p(m)}$ the space of all measurable functions $f : S \mapsto \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that :

$$\int_S |f(t)|^p dm(t) < +\infty,$$

and we put:

$$\boxed{\|f\|_p = \Big(\int_S |f(t)|^p dm(t)\Big)^{\frac{1}{p}}}.$$

Note that $\|f\|_p = 0$ if and only $f = 0$ m-almost everywhere.

Theorem 1.1.1 (Minkowki Inequality). *Set $1 \leq p < \infty$. for $f, g \in \mathscr{L}^p(m)$, we have the* ***Minkowki Inequality*** *:*

$$\boxed{\left(\int_S |f+g|^p dm\right)^{1/p} \leqslant \left(\int_S |f|^p dm\right)^{1/p} + \left(\int_S |g|^p dm\right)^{1/p}}$$

It follows that $\mathscr{L}^p(m)$ is a vector subspace of the space of measurable functions and that $\|.\|_p$, is a semi-norm on $\mathscr{L}^p(m)$. For $p = 1$, the inequality is obvious.

Proof. The proof is the same as for the sequences. We place ourselves in the case $p > 1$. We can assume $\|f\|_p > 0$ and $\|g\|_p > 0$(because otherwise $f = 0$ m-a.e. and then $f + g = g$ m-a.e, or $g = 0$ m-a.e and then $f + g =$ m-a.e). We apply the convexity inequality $[\alpha u + (1-\alpha)v]^p \leqslant \alpha u^p + (1-\alpha)v^p$ with $\alpha = \|f\|_p / (\|f\|_p + \|g\|_p) \in [0, 1]$, $u = |f(t)|/\|f\|_p$ and $v = |g(t)|/\|g\|_p$. Since $\alpha/\|f\|_p = (1-\alpha)/\|g\|_p = 1/\left(\|f\|_p + \|g\|_p\right)$, we have $\left(\frac{|f(t)|+|g(t)|}{\|f\|_p+\|g\|_p}\right)^p \leqslant$

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For all $x_1, x_2, \ldots, y_1, y_2, \ldots \in \mathbf{K}$. We obtain it from the previous one by making the number of terms tend towards infinity: for all $N \geqslant 1$, we have: $\left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{1/p} \leqslant$

$\left(\sum_{n=1}^{N} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{N} |y_n|^p\right)^{1/p} \leqslant (\sum_{n=1}^{\infty} |x_n|^p)^{1/p} + (\sum_{n=1}^{\infty} |y_n|^p)^{1/p}$

Hölder's inequality is generalized in the same way. If $1 < p < \infty$ and if q is the conjugate exponent of p, we have:

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In particular, when $x = (x_n)_n \in \ell_p$ and $y = (y_n)_n \in \ell_q$, we have $xy \in \ell_1$ and $\|xy\|_1 \leq \|x\|_p \|y\|_q$.

The spaces ℓ_p are in fact special cases of the Lebesgue spaces $L^p(m)$, whose definition we will recall below, corresponding to the counting measure on N^* .

Function spaces

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that verifies $|f|_1 \leq |f|_\infty$.

c) On the space $\mathscr{C}([0, 1])$ of functions k times continuously derivable on $[0, 1]$, the norm can be set as fellown:

$$\|f\|^{(k)} = \max\left\{\|f\|_\infty, \|f'\|_\infty, \ldots, \left\|f^{(k)}\right\|_\infty\right\}$$

2) Lebesgue spaces.

Let $(S, \mathscr{F}, m)$ be a measured space; for $1 < p < \infty$, we denote $\boxed{\mathscr{L}^p(m)}$ the space of all measurable functions $f : S \mapsto \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that :

$$\int_S |f(t)|^p dm(t) < +\infty,$$

and we put:

$$\boxed{\|f\|_p = \Big(\int_S |f(t)|^p dm(t)\Big)^{\frac{1}{p}}}.$$

Note that $\|f\|_p = 0$ if and only $f = 0$ m-almost everywhere.

Theorem 1.1.1 (Minkowki Inequality). *Set $1 \leq p < \infty$. for $f, g \in \mathscr{L}^p(m)$, we have the **Minkowki Inequality** :*

$$\boxed{\left(\int_S |f+g|^p dm\right)^{1/p} \leqslant \left(\int_S |f|^p dm\right)^{1/p} + \left(\int_S |g|^p dm\right)^{1/p}}$$

It follows that $\mathscr{L}^p(m)$ is a vector subspace of the space of measurable functions and that $\|.\|_p$, is a semi-norm on $\mathscr{L}^p(m)$. For $p = 1$, the inequality is obvious.

Proof. The proof is the same as for the sequences. We place ourselves in the case $p > 1$. We can assume $\|f\|_p > 0$ and $\|g\|_p > 0$(because otherwise $f = 0$ m-a.e. and then $f + g = g$ m-a.e, or $g = 0$ m-a.e and then $f + g =$ m-a.e). We apply the convexity inequality $[\alpha u + (1-\alpha)v]^p \leqslant \alpha u^p + (1-\alpha)v^p$ with $\alpha = \|f\|_p / (\|f\|_p + \|g\|_p) \in [0, 1]$, $u = |f(t)|/\|f\|_p$ and $v = |g(t)|/\|g\|_p$. Since $\alpha/\|f\|_p = (1-\alpha)/\|g\|_p = 1/\left(\|f\|_p + \|g\|_p\right)$, we have $\left(\frac{|f(t)|+|g(t)|}{\|f\|_p+\|g\|_p}\right)^p \leqslant$

$\frac{\alpha}{\|f\|_p^p}|f(t)|^p + \frac{1-\alpha}{\|g\|_p^p}|g(t)|^p$, hence, by integrating:

$$\int_S \frac{(|f(t)| + |g(t)|)^p}{(\|f\|_p + \|g\|_p)^p} dm(t) \leqslant \frac{\alpha}{\|f\|_p^p} \int_S |f(t)|^p dm(t) + \frac{1-\alpha}{\|g\|_p^p} \int_S |g(t)|^p dm(t)$$
$$= \alpha + (1-\alpha) = 1$$

this gives the result since $|f(t) + g(t)| \leq |f(t)| + |g(t)|$. □

We saw that $\|.\|_p$, is not a norm in general, since $\|f\|_p = 0$ if and only $f = 0$ m-almost everywhere. If$\mathscr{N}$ denotes the space of measurable functions $f : S \mapsto \mathbb{K}$ null m-almost everywhere, the quotient space $\boxed{L^p(m) = \mathscr{L}^p(m)/\mathscr{N}}$ is then normed if we set $\|\tilde{f}\|_p = \|f\|_p$.
In practice, we will not distinguish between the function and its m-equivalence class almost everywhere $\tilde{f}$, and we will therefore write $f \in L^p(m)$ instead of $f \in \mathscr{L}^p(m)$. However, sometimes one have to be careful, especially when handling non-countable quantities of functions. This distinction may already occur for questions of measurability. We can also see this in the following example:
Let $\mathscr{F}$ be the set of all finite parts of $[0, 1]$; for all $A \in \mathscr{F}$, we have, in terms of the Lebesgue measure, $\mathbf{1}_A = 0$ a.e.; so $\tilde{\mathbf{1}_A} = \tilde{0}$. But, on the other hand, $sup_{A \in \mathscr{F}} \mathbf{1}_A(x) = 1$ for all $x \in [0, 1]$; so $(sup_{A \in \mathscr{F}} \tilde{\mathbf{1}}_A) = \tilde{\mathbf{1}}$.
As mentionned for sequences, Hölder's inequality is very useful.

Theorem 1.1.2 (Hölder Inequality). *If $1 < p < \infty$ and if q is the conjugate exponent of p, then we have, for $f \in \mathscr{L}^P(m)$and $g \in$ $mathscr L^q(m)$, Hölder's inequality:*

$$\boxed{\int_S |fg|^p dm \leqslant \left(\int_S |f|^p dm\right)^{1/p} \left(\int_S |g|^p dm\right)^{1/p}}$$

For $p = q = 2$, we call it Cauchy-Schwarz inequality:

$$\boxed{\int_S |fg|^2 dm \leqslant \left(\int_S |f|^2 dm\right)^{1/2} \left(\int_S |g|^2 dm\right)^{1/2}},$$

if $f, g \in \mathscr{L}^2(m)$. (It was demonstrated by Bouniakowski in 1859 and re-proven by Schwarz in 1885; it generalizes the inequality for sums demonstrated by Cauchy). It is demonstrated in the same way as for sums, by integrating instead of adding.

Proof. We can assume $\|f\|_p > 0$ et $\|g\|_q > 0$ because otherwise $f = 0$ $m - a.e.$ or $g = 0$ $m - a.e.$, and then $fg = 0m - a.e.$ We use the inequality $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$ with $a = |f(t)|/\|f\|_p$ and $b = |g(t)|/\|g\|_q$. By integrating, we obtain:

$$\int_S \frac{|f(t)g(t)|}{\|f\|_p\|g\|_q} dm(t) \leqslant \frac{1}{p}\int_S \frac{|f(t)|^p}{\|f\|_p^p} dm(t) + \frac{1}{q}\int_S \frac{|g(t)|^q}{\|g\|_q^q} dm(t) = \frac{1}{p} + \frac{1}{q} = 1$$

hence the result. □

As an application we have the following result.

Proposition 1.1.4. *Let $(S, \mathscr{F}, m)$be a measured space of finite measure. Then, for $1 < p_1 < p_2 < \infty$, we have $\mathscr{L}^{p_2}(m) \subseteq \mathscr{L}^{p_1}(m) \subseteq \mathscr{L}^(1(m)$. Moreover if $m(S) = 1$ (i.e. m is a probability measure), then $\|f\|_1 < \|f\|_{p_1} < \|f\|_{p_2}$ for all $f \in \mathscr{L}^{p_2}(m)$.*

Proof. We can assume $p_1 < p_2$ Let $p = \frac{p_2}{p_1}$. As $p > 1$, we can use Hölder's inequality:

$$\int_S |f|^{p_1} dm \leqslant \left(\int_S 1^q dm\right)^{1/q} \left(\int_S (|f|^{p_1})^p \, dm\right)^{1/p} = [m(S)]^{1/q} \left(\int_S |f|^{p_2} dm\right)^{1/p};$$

hence $\|f\|_{p_1} \leq [m(S)]^{\frac{1}{p_1} - \frac{1}{p_2}} \|f\|_{p_2}$.

The second inclusion is obtained by replacing p_2 by p_1 and taking $p_1 = 1$. □

Remark 1. *On the contrary, for spaces ℓ_p, we have the opposite inclusions;*

for $1 < p_1 < p_2 < \infty$:

$$\ell_1 \subseteq \ell_{p_1} \subseteq \ell_{p_2} \subseteq c_0 \subseteq \ell_\infty.$$

In addition, $\|x\|_\infty \leq \|x\|_{p_2} \leq \|x\|_{p_1} \leq \|x\|_1$ for all $x \in \ell_1$.

Indeed, if $x \in \ell_{p_2}$,$\sum_{n=1}^{\infty} |x_n|^{p_2} < +\infty$; so $x_n \underset{n\to\infty}{\longrightarrow} 0$.

Moreover, for all $n \geq 1$, $|x_n| \leqslant (\sum_{n=1}^{\infty} |x_n|^{p_2})^{1/p_2} = \|x\|_{p_2}$; therefore $\|x\|_\infty = \sup_{n\geqslant 1} |x_n| \leqslant \|x\|_{p_2}$. Now, if $x \in \ell_{p_1}$ is not zero, let's set $x' = x/\|x\|_{p_1}$. We have $\|x'\|_{p_1} = 1$, that is to say $\sum_{n=1}^{\infty} |x'_n|^{p_1} = 1$. It follows that $|x'_n|^{p_1} \leqslant 1$, and therefore

$|x'_n| \leqslant 1$, for all $n \geqslant 1$. Then, for all $n \geqslant 1, |x'_n|^{p_2} \leqslant |x'_n|^{p_1}$, since $p_2 \geqslant p_1$. It follows that $\sum_{n-1} |x'_n|^{p_2} \leqslant \sum_{n-1} |x'_n|^{p_1} = 1$, that is, $\sum_{n-1} |x_n|^{p_2} \leqslant \|x\|_{p_1}^{p_2}$. So $x \in \ell_{p_2}$ and $\|x\|_{p_2} \leqslant \|x\|_{p_1}$.

Remark 2. *On the other hand, it is important to note that $\boxed{\mathscr{L}^{p_1}(\mathbb{R}) \nsubseteq \mathscr{L}^{p_2}(\mathbb{R})}$ for all $p_1 \notin p_2$.*

Indeed, if $p_1 < p_2$, the function defined by $f(t) = 1/t^{1/p_2}$ for $0 < t \leqslant 1$, and by $f(t) = 0$ elsewhere, is in $\mathscr{L}^{p_1}(\mathbb{R})$ because $p_1/p_2 < 1$, but not in $\mathscr{L}^{p_2}(\mathbb{R})$. If $p_1 > p_2$, then the function defined by $f(t) = 1/t^{1/p_2}$ for $t \geqslant 1$, and $f(t) = 0$ for $t < 1$, is in $\mathscr{L}^{p_1}(\mathbb{R})$ because this time $p_1/p_2 > 1$, but is not in $\mathscr{L}^{p_2}(\mathbb{R})$.

1.1.4 Equivalent norms

Definition 1.1.4. *Let E be a vector space with two norms $\|.\|$ and $|||.|||$. We say that $|||.|||$is finer than $\|.\|$(and that $\|.\|$ is less fine than $|||.|||$) if it exists a constant $K > 0$ such that:*

$$\|x\| \leq K \, |||x|||, \quad \forall x \in E.$$

This is equivalent to saying that the identity application:

$$id_E : (E, |||.|||) \to (E, \|.\|)$$

is continuous.

This is still equivalent to saying that:

$$B_{|||.|||}(0, r/K) \subseteq B_{\|.\|}(0, r);$$

the balls for $|||.|||$ are therefore 'smaller' than the balls for $\|.\|$: they separate the points better; more precisely, the topology defined by $|||.|||$ is finer than that defined by $\|.\|$ (there are more open sets).

Exemple. In $\mathscr{C}([0,1])$, the norm $\|.\|_\infty$ is finer than the norm $\|.\|_1$.

Definition 1.1.5. *We say that two norms $\|.\|$ and $|||.|||$ on the vector space E are equivalent if there exist two constants $K_1, K_2 > 0$ such that:*

$$K_1\|x\| \leq |||x||| \leq K_2\|x\|, \quad \forall x \in E.$$

In other words, each is thinner than the other.

This amounts to saying that **the identity application** *I_d carries out an isomorphism of E on itself (or rather of E equipped with $\|.\|$ on E equipped with $|||.|||$). This also means that $\|.\|$and $|||.|||$. define the same topology on E.*

Exemples.

1) In $\mathbb{K}^n$ the norms $\|.\|_p$ for $1 \leq p \leq \infty$ are equivalent:

$$\|x\|_\infty \leq \|x\|_p \leq \|x\|_1 \leq n\|x\|_\infty.$$

We will see that in fact all the norms on $\mathbb{K}^n$ are equivalent to each other.

2) In $\mathscr{C}([0,1])$, the norms $\|.\|_\infty$ and $\|x\|_1$ are not equivalent.

1.2 Banach spaces

1.2.1 Cauchy sequences

Definition 1.2.1. *A sequence $(x_k)_k$ of elements of a normed space E is called a Cauchy sequence if:*

$$(\forall \varepsilon > 0) \quad (\exists N \geqslant 1) \quad k, l \geqslant N \quad \Longrightarrow \quad \|x_k - x_l\| \leqslant \varepsilon.$$

Any convergent sequence is Cauchy sequence.

Definition 1.2.2. *We say that a normed space is complete if every Cauchy sequence is convergent. We call Banach space any complete normed space.*

Exemples.

a) It is immediate to see that $\ell_p^n = ((K)^n, \|.\|_p)$ is complete for $1 \leq p \leq\ infty$.

b) The spaces c_0 and ℓ_p, for $1 \leq p \leq \infty$ are complete .

c) $(\mathscr{C}(K), \|.\|_\infty)$ is complete: any uniformly Cauchy sequence is uniformly convergent, and if they are continuous, the limit is continuous too.

   On the other hand, $(\mathscr{C}([0,1]), \|.\|_1)$ is not complete.

d) $(\mathscr{C}^k([0,1]), \|.\|_\infty)$ is not complete for $k \geq 1$, but $(\mathscr{C}^k([0,1]), \|.\|^{(k)})$ is complete.

e) Lebesgue spaces are complete. This is the subject of the following theorem.

Theorem 1.2.1 (Riesz-Fisher theorem). *For any measured space $(S, \mathscr{T}, m)$, and for $1 \leq p < \infty$, the space $L^p(m)$ is a Banach space.*

E. Fisher et. F. Riesz actually demonstrated, independently, in 1907 that $L^2([0,1])$ is isomorphic to ℓ_2; this essentially relies on the fact that $L^2([0,1])$ is complete (see Chapter 2 on Hilbert spaces); this is why we give the name Riesz-Fisher to this theorem, proven in fact, for $L^p([0,1])$ and $1 < p < \infty$, by F. Riesz in 1910 (and to distinguish it from the many other theorems due to F. Riesz).

proof of lemma. Let $(F_n)_n$ be a Cauchy sequence in $L^P(m)$. Let's choose a representative $f_n \in \mathscr{L}^p(m)$ de F_n.

a) As the sequence is Cauchy, we can construct a subsequence $(f_{n_k})_j$ with $(n_1 < n_2 < \ldots)$ such that:

$$\|f_{n_{k+1}} - f_{n_k}\|_p \leqslant \frac{1}{2^k} \quad \forall k \geqslant 1.$$

Let's put:

$$\begin{cases} g_k = \sum_{j=1}^{k} \left|f_{n_{j+1}} - f_{n_j}\right| \\ g = \sum_{j=1}^{\infty} \left|f_{n_{j+1}} - f_{n_j}\right| \end{cases}$$

These functions are measurable and we have:

$$\|g_k\|_p \leqslant \sum_{j=1}^{k} \left\|\left|f_{n_{j+1}} - f_{n_j}\right|\right\|_p = \sum_{j=1}^{k} \left\|f_{n_{j+1}} - f_{n_j}\right\|_p \leqslant \sum_{j=1}^{k} \frac{1}{2^j} \leqslant 1.$$

Fatou's Lemma, applied to the sequence $(g_k^p)_{k\geqslant 1}$, gives:

$$\int_S g^p dm \leqslant \liminf_{k\to\infty} \int_S g_k^p dm = \liminf_{k\to\infty} \|g_k\|_p^p \leqslant 1.$$

The function g^p is therefore integrable. In particular it is finished almost everywhere; so g too. This means that the series $\sum_{k\geq 1}(f_{n_{k+1}}(t) - f_{n_k}(t))$ converges absolutely, for almost all $t \in S$.

Hence, let's put:

$$f(t) = \begin{cases} f_{n_1}(t) + \sum_{k=1}^{\infty} \left(f_{n_{k+1}}(t) - f_{n_k}(t)\right) & \text{si } g(t) < +\infty \\ 0 & \text{otherwise} \end{cases}$$

Then f is measurable and:

$$f(t) = \lim_{k\to\infty} f_{n_k}(t) \quad \forall t \in S \textit{ almost everywhere}.$$

b) It remains to be seen that the sequence is from cauchy, there exists an integer $N \geq 1$ such that:

$$n, k \geqslant N \quad \Longrightarrow \quad \|f_n - f_k\|_p \leqslant \varepsilon.$$

For $k \geqslant N$, Fatou's Lemma gives:

$$\int_S |f - f_k|^p \, dm \leqslant \liminf_{j\to\infty} \int_S \left|f_{n_j} - f_k\right|^p dm \leqslant \varepsilon^p.$$

We first deduce that $(f - f_k) \in \mathscr{L}^p(m)$, therefore that $f = (f - f_k) + f_k \in \mathscr{L}^p(m)$; and then, since $\varepsilon > 0$ is arbitrary, that $\lim_{k\to\infty} \|f - f_k\|_p = 0$.

c) Finally, if we write $F \in L^p(m)$ for the m-almost everywhere equivalence class of f, we have $\lim_{k\to\infty} \|F - F_k\|_p = \lim_{k\to\infty} \|f - f_k\|_p = 0$.

■

Remark. It is worth mentioning that the following underlined and <u>very important</u> result has been demonstrated (we will no longer make a distinction between a function and its equivalence class almost everywhere):

Theorem 1.2.2. *If $f_{n} \underset{n\to\infty}{\mapsto} f$ in $L^p(m)$, with $1 \leqslant p < \infty$, then we can extract a sub- sequence $(f_{n_k})_k$ which converges almost everywhere to f.*

Remarks. a) It's possible that the sequence itself will not converge anywhere. For example, on $S =]0,1]$, i.e. $f_n = \mathbf{I}_{]\frac{1}{2^k},\frac{l+1}{2^k}]}$ when $n = 2^k + l, 0 \leqslant l \leqslant 2^k - 1$:

Hence, $\|f_n\|_p = \frac{1}{2^{k/p}}$ for $2^k \leqslant n \leqslant 2^{k+1} - 1$; it fellows that, $f_n \underset{n\to\infty}{\longrightarrow} 0$ in $L^p([0,1])$, but for no $t \in]0.1[$, the sequence $(f_n(t))_n$ is convergent. However, the sub-sequence $(f_{2k})_{k \geqslant 0}$, for example, pointwise convergences a.e.

b) It may be noted that $\mathscr{C}[-1,1]$ is not a closed subspace in $L_2[-1,1]$.

c) The space $\mathscr{C}(\Omega)$ is a dense subspace of $L_2(\Omega)$.

d) The set of all polynomials is dense in $L_2(\Omega)$.

1.3 Normed finite-dimensional vector spaces

1.3.1 Equivalence of norms

Theorem 1.3.1. *On a finite-dimensional vector space, all norms are equivalent to each other.*

Proof. 1) Let $\|.\|$ be an arbitrary norm on E. We will show that it is equivalent to a particular norm on E, so that, by transitivity, two arbitrary norms will be equivalent.

2) Let $\{e_1, \ldots, e_n\}$ be a base of E. If $x = \sum_{k=1}^{n} \xi_k e_k$, we set:

$$|\|x\,|\,\| = \max\{|\xi_1|, \ldots, |\xi_n|\}.$$

Thus, $(E, |||\cdot|||)$ is isometric to $(\mathbb{K}^n, \|\cdot\|_\infty) = \ell_\infty^n(\mathbb{K})$, by the application

$$V: \quad \mathbb{K}^n \longmapsto E$$
$$a = (a_1, \ldots, a_n) \quad \longmapsto V(\xi) = \sum_{k=1}^{n} a_k e_k.$$

We also have:

$$\|x\| \leqslant \sum_{k=1}^{n} |\xi_k| \, \|e_k\| \leqslant \left(\sum_{k=1}^{n} \|e_k\| \right) \cdot \max_{1 \leqslant k \leqslant n} |\xi_k| = K|||x|||.$$

3) This means that the identity mapping $id_E : (E, |||.|||) \to (E, \|\cdot\|)$ is continuous . So, the application:

$$N: \quad (E, |||\cdot|||) \longrightarrow \mathbb{R}_+$$
$$x \longmapsto \|x\|$$

is also continuous, by the Proposition 1.1.1

4) Or:

$$S_n = \{a = (a_1, \ldots, a_n) \in \mathbf{K}^n; \|a\|_\infty = 1\}.$$

It is a closed and bounded, therefore compact, part of $\mathbf{K}^n$ (note that the norm $\|\cdot\|_\infty$ defines the usual topology on $\mathbf{K}^n$). so :

$$S = \{x \in E; |||x||| = 1\}$$

is a compact part of $(E, |||.|||)$ (by isometry: $S = V(S_\infty)$).

5) It follows that there exists $x_0 \in S$ such that $\|x_0\| = N(x_0) = \inf_{x \in S} N(x) = \inf_{x \in S} \|x\|$. Since $x_0 \neq 0$ (since $|||x_0\| = 1$), we have $c = \|x_0\| > 0$. It means that :

$$(\forall x \in S) \quad \|x\| \geqslant c.$$

By homogeneity (for all $x \neq 0, x' = x/|||x||| \in S$), we obtain:

$$(\forall x \in E) \quad \|x\| \geqslant c|||x|||,$$

□

This was to be demonstrated.

Remark. In passing, we showed:

Corollary 1.3.1. *Any finite-dimensional normed space n is isomorphic to $\mathbb{K}^n$, equiped with one of its usual norms.*

It follows:

Corollary 1.3.2. *If E is a finite-dimensional normed space, its bounded closed parts are compact.*

Corollary 1.3.3. *1) Any finite-dimensional normed space is complete.*

2) Any vector subspace of finite dimension in a normed space is closed in this space.

3) If E is a finite-dimensional normed space, then any linear mapping $T : E \to F$ in an arbitrary normed space F is continuous.

Proof. 1) follows immediately from the Corollary 1.3.1, and 2) from the fact that everything under complete space is closed. For 3), it suffices to notice that if $e_1, \ldots, e_n$ is a base of E, and $|||.|||$ the associated norm as in the proof of the Theorem 1.3.1, then, for $x = \sum_{k=1}^{n} a_k e_k \in E$, we have:

$$\|T(x)\|_F \leqslant \left(\sum_{k=1}^{n} \|T(e_k)\|_F\right) \max_{k \leqslant n} |a_k| = C \mid \|x\| \leqslant CK\|x\|_E$$

since $|||.|||$ and $\|.\|_E$ are equivalent. □

We will be careful, on the other hand, that if it is the arrival space which is of finite dimension, continuity is not automatic (since there exist non-continuous linear forms, if E is of infinite dimension.

1.3.2 Compactness of the balls

We have seen in the proof of Theorem 1.3.1 that the essential point (via the Corollary 1.3.2) is that the closed bounded parts of a finite-dimensional normed space are compact. Note that it is equivalent to saying that all closed balls are compact. We will see that this actually only happens in finite dimension.

Theorem 1.3.2 (Riesz Theorem, 1918). *If a normed space E has a compact ball $B(x_o, r)$, of radius $r > 0$, then it is of finite dimension.*

We deduce that in an infinite dimensional space, the compacts are 'very thin':

Corollary 1.3.4. *If E is a normed space of infinite dimension, then every compact of E has an empty interior.*

Indeed, if K is a non-empty interior compact, it contains a closed ball of radius $r > 0$, which is therefore compact, and therefore E is of finite dimension.

Note that if a ball is compact, it is necessarily a closed ball. On the other hand, if a ball, of radius $r > 0$, is compact, then all closed balls are, since they are homeomorphic to each other (those of zero radius being compact anyway). It is therefore sufficient to show that if E is of infinite dimension, then its unit ball B_E is not compact. To do this, we will use a lemma.

Lemma 1.3.1 (Riesz's lemma). *Let F be a closed vector subspace of a normed space E, which is not an entire E. Then, for any number δ such that $0 < \delta < 1$, there exists $x \in E$ such that:*

$$\begin{cases} \|x\| = 1 \\ \operatorname{dist}(x, F) \geqslant 1 - \delta \end{cases}$$

Let's remember that :

$$\operatorname{dist}(x, F) = \inf_{y \in F} \|x - y\|.$$

If F is of finite dimension, an argument of compactness makes it possible to show that in fact we can choose such a $x \in E$, of norm 1, with dist $(x, F) = 1$, but we won't need it. In the case of the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$, it suffices to take x of norm 1 and orthogonal a F (because then, for all $y \in F$, we have $\|x-y\|_2^2 = \|x\|_2^2 + \|y\|_2^2$, by the Pythagorean Theorem; therefore dist $(x, F) \geqslant \|x\|_2 = 1$, hence the equality because $|x\| \geqslant$ dist (x, F), since $0 \in F$). This is why this lemma is sometimes called the Quasi-Perpendicular Lemma.

proof of Riesz Theorem. Let E be a normed space of infinite dimension. Let's set a number $\delta \in]0,1[$,; for example $\delta = \frac{1}{2}$.

Let us start from a $x_1 \in E$, of norm 1, and take for F the vector subspace F_1 generated by x_1. As it is of dimension 1, it is closed, and is not equal to E, since E is of infinite dimension. The lemma gives a $x_2 \in E$, of norm 1 such that:

$$\|x_2 - x_1\| \geq dist(x_2, F_1) \geq \frac{1}{2}.$$

Let us then take for F the vector subspace F_2 generated by x_1 and x_2 . It is of dimension 2 (because $x_2 \notin F_1$), and is therefore closed, and different from E; there therefore exists $x_3 \in E$, of norm 1 such that:

$$\|x_3 - x_1\| et \|x_3 - x_2\| \geq dist(x_3, F_2) \geq \frac{1}{2}.$$

As E is of infinite dimension, we can iterate the process indefinitely. We obtain a sequence $(x_k)_{k\geq 1}$ of vectors of norm 1 such that:

$$\|x_k - x_l\| \geq \frac{1}{2}, \quad \forall k \neq l.$$

This sequence cannot have any convergent subsequence. As it is contained in the unit ball of E, this ball is not compact. ∎

Remark. We have in fact demonstrated a little more than what was stated, namely that if E is of infinite dimension, its unit sphere S_E is not compact (note that S_E is closed in B_E; so if B_E is compact, so is S_E).

proof of lemma. Like $F \neq E$, we can find $x_0 \in E$ such that $x_0 \notin F$. As F is closed, we have:

$$d = dist(x_0, F) > 0.$$

As $0 < \delta < 1$, we have $\frac{d}{1-\delta} > d$ and we can therefore find a $y_0 \in F$ such that $\|x_0 - y_0\| \leq \frac{d}{(1-\delta)}$. All that remains is to 'correct' x_o by y_0 and to normalize this vector: let $x = \frac{x_0 - y_0}{\|x_0 - y_0\|}$; it is indeed a vector of norm 1 and, like $y_0 + \|x_0 - y\| y \in F$,

we have :

$$\begin{aligned}\|x - y\| &= \frac{1}{\|x_0 - y_0\|} \|(x_0 - y_0) - \| x_0 - y_0\| y\| \\ &\geqslant \frac{1}{\|x_0 - y_0\|} \operatorname{dist}(x_0, F) = \frac{d}{\|x_0 - y_0\|} \geqslant 1 - \delta,\end{aligned}$$

for all $y \in F$. ■

1.4 Linear mappings

For linear mapings, we have a very simple and very useful criterion of continuity.

Proposition 1.4.1. *Let* $(E, \|.\|_E)$ *and* $(F, \|.\|_F)$ *be two normed spaces and let* $T : E \mapsto F$ *be a linear map. Then* T *is continuous if and only if there exists a constant* $K \geq 0$ *such that:*

$$\boxed{\|T(x)\|_F \leq K \|x\|_E, \quad \forall x \in E.}$$

Proof. It is clear that this property causes the continuity of T because we have, thanks to linearity:

$$\|T(x) - T(y)\|_F \leq K \|x - y\|_E, \quad \forall x, y \in E;$$

T is therefore even Lipschitzian.

Conversely, if T is continuous at 0, we have, by definition:

$$(\exists K > 0) \quad \|y - 0\|_E = \|y\|_E \leqslant 1/K \quad \Longrightarrow \|T(y)\|_F = \|T(y) - T(0)\|_F \leqslant 1$$

For all $x \in E$, non-zero, let $y = \frac{1}{K\|x\|_E} x$; we have $\|y\|_E = 1/K$ and the implication above gives, thanks to the homogeneity of T and the norm:

$$\frac{1}{K\|x\|_E}\|T(x)\|_F \leqslant 1,$$

hence $\|T(x)\|_F \leqslant K\|x\|_E$. As this inequality is obviously true for $x = 0$, this shows Proposition 1.4.1 .

We therefore have $\sup_{x\neq 0} \frac{\|T(x)\|_F}{\|x\|_E} < +\infty$. The following proposition is then obvious: □

Proposition 1.4.2. *Let $T : E \to F$ be a continuous linear mapping. If we put* $\boxed{\|T\| = \sup_{x\neq 0} \frac{\|T(x)\|_F}{\|x\|_E}}$, *then:*

$$\boxed{\|T(x)\|_F \leqslant \|T\|\|x\|_E, \quad \forall x \in E}$$

$\|T\|$ is therefore the smallest constant $K \geqslant 0$ appearing in the Proposition1.4.1.

Proposition 1.4.3. *We also have*

$$\boxed{\|T\| = \sup_{\|x\|_E\leq 1} \|T(x)\|_F = \sup_{\|x\|_E=1} \|T(x)\|_F.}$$

Proof. Let's call S the first expression and S_1 the next one. We of course have $S_1 \leq S$, and also $S \leq \|T\|$, since $\|T(x)\|_F \leq \|T\|$ if $\|x\|_E \leq 1$, by definition of $\|T\|$. It remains to be seen that $\|T\| \leq S_1$; but :

$$\|T\| = \sup_{x\notin 1} \frac{\|T(x)\|_F}{\|x\|_E} = \sup_{x\notin 1} \|T(\frac{x}{\|x\|_E})\|_F \leq S_1,$$

Because $\frac{x}{\|x\|_E}$ is of norm 1. □

Proposition 1.4.4. *Let $\mathscr{L}(E,F)$ be the space of all continuous linear mappings of E in F. The mapping $T \mapsto \|T\|$ is a norm on $\mathscr{L}(E,F)$, called the operator norm.*

If F is complete, so is $\mathscr{L}(E,F)$.

Proof. The fact that this is a norm is easy to verify.

Let F be complete, and let $(T_n)_n$ be a Cauchy sequence in $\mathscr{L}(E,F)$. Then, for all $x \in E$, the sequence $(T_n(x))_n$, is Cauchy in F, by virtue of the inequality:

$$\|T_n(x) - T_k(x)\|_F \leqslant \|T_n - T_k\| \|x\|_E, \tag{1.1}$$

It therefore converges to an element $T(x) \in F$. It is easy to see that then $T : E \to F$ is linear. It is continuous because:

$$\|Tx\|_F = \lim_{n\to\infty} \|T_n x\|_F \leqslant \limsup_{n\to\infty} \|T_n\| \, \|x\|_E \leqslant \left(\sup_{n\geqslant 1} \|T_n\|\right) \|x\|_E$$

and because $(\sup_{n\geqslant 1} \|T_n\|) < +\infty$ since any Cauchy sequence is bounded. Finally, by making k tend towards infinity in 1.1, we obtain:

$$\|T_n(x) - T(x)\|_F \leqslant \left(\limsup_{k\to\infty} \|T_n - T_k\|\right) \|x\|_E$$

when n tends to infinity, since $(T_n)_n$ is Cauchy. So $(T_n)_n$ converges to T for the operator norm.

In particular, if $F = \mathbb{K}, \mathscr{L}(E, \mathbb{K})$ is always complete. □

Definition 1.4.1. *If the linear mapping $T : E \mapsto F$ is bijective continuous* *and* *if $T^{-1} : F \mapsto E$ is continuous, we say that T is an isomorphism (of normed spaces) between E and F.*

We say that E and F are isomorphic if there exists an isomorphism between E and F; we says that they are isometric if there exists an isometric isomorphism $T : E \to F$.

Note that saying that a mapping $T : E \to F$ is isometric means that we have $\|T(x_1) - T(x_2)\|_F = \|x_1 - x_2\|_E$ for all $x_1, x_2 \in E$. When T is linear, this is expressed by $\|T(x)\|_F = \|x_e\|_E$ for all $x \in E$;T is therefore in particular of norm $\|T\| =1$. All isometry is injective; to say that it is bijective is therefore equivalent to saying that it is surjective. In this case, T^{-1} is also an isometry; it is therefore automatically continuous.

Saying that T is an isomorphim means that T is bijective linear and that there exist two *constants* $0 < \alpha < \beta < \infty$ *such that:*

$$\boxed{\alpha\|x\|_E \leq \|Tx\|_F \leq \beta\|x\|_E} \quad \text{for all } x \in E$$

Indeed, if T *is an isomorphism, the continuity of* T^{-1} *allows us to write:*
$\|T^{-1}y\|_E \leq \|T^{-1}\|\|y\|_F$ *for all* $y \in F$ *, i.e.* $\|x\|_E \leq \|T^{-1}\|\|Tx\|_F$ *for all* $x \in E$*. We therefore have the double inequality, with* $\alpha = \frac{1}{\|T^{-1}\|}$ *and* $\beta = \|T\|$ *. Conversely, if we have this double inequality, then* T *is continuous and* $\|T\| \leq \beta$ *and* T^{-1} *is continuous and* $\|T^{-1}\| \leq \frac{1}{\alpha}$*, since* $\alpha\|T^{-1}y\|_E \leq \|y\|_E$ *for all* $y \in F$*.*

We can also notice that the left inequality results in the injectivity of T*.*

1.5 Dual of vector normed space

Definition 1.5.1. $\mathscr{L}(E, \mathbb{K})$ *is denoted by* E^* *and is called the dual of* E*. It is still a Banach space.*

Note that E^* is the topological dual of E, and is strictly smaller than the algebraic dual - the space of all linear functionals -, of E, at least if E is infinite dimension. The norm of $\varphi \in E^*$ is therefore defined by:

$$\boxed{\|\varphi\| = \|\varphi\|_{E^*} = \sup_{x \neq 0} \frac{|\varphi(x)|}{\|x\|} = \sup_{\|x\| \leqslant 1} |\varphi(x)| = \sup_{\|x\|=1} |\varphi(x)|.}$$

Notation. We often use the notation $\boxed{\langle \varphi, x \rangle = \varphi(x)}$.

Theorem 1.5.1. *The dual space* X' *of a normed space* X *is a Banach space (whether or not* X *is).*

Remark.

- Other terms are dual, adjoint space and conjugate space.

- Algebraic dual space X^* of X is the vector space of all linear functionals on X.

Example 1.1. • *Space* $\mathbb{R}^n$*: The dual space of* $\mathbb{R}^n$ *is* $\mathbb{R}^n$*.*