⑤ Notice that $B=\{\frac{1}{x^2+1} \mid x\in\mathbb{R}\}$ is not empty (because $1=\frac{1}{0+1}\in B$)

Now, we have: $\forall x\in\mathbb{R}$: $x^2\geq 0$, which yields that: $0<\frac{1}{x^2+1}\leq 1$

So, the set B is Bounded from above by 1 and since $1\in B$, we conclude

that: $\mathrm{Max}(B)=1=\mathrm{Sup}(B)$.

We have 0 is a lower bound of B and $0\notin B$, then $\mathrm{Min}(B)\neq 0$.

to prove that: $\inf(B)=0$, we use the characteristic property of Inf.

that is: $\forall\varepsilon>0, \exists b\in B: 0<b<0+\varepsilon$, or equivalently, that:

$$\forall\varepsilon>0, \exists x\in\mathbb{R}: \quad 0<\frac{1}{x^2+1}<\varepsilon.$$

On the other hand:

$$\frac{1}{x^2+1}<\varepsilon \iff x^2>\frac{1}{\varepsilon}-1.$$

thus, if $\varepsilon\geq 1$ (i.e., $\frac{1}{\varepsilon}-1\leq 0$), then we can choose any $x\in\mathbb{R}^*$.

If $\varepsilon<1$ (i.e., $\frac{1}{\varepsilon}-1>0$), then $(x^2>\frac{1}{\varepsilon}-1) \iff \left(x>\sqrt{\frac{1}{\varepsilon}-1} \text{ or } x<\sqrt{\frac{1}{\varepsilon}-1}\right)$

in this case, it is sufficient to choose x from the interval $]-\infty,-\sqrt{\frac{1}{\varepsilon}-1}[$

or $]\sqrt{\frac{1}{\varepsilon}-1},+\infty[$.

Exercise 3

Part ①

ⓐ Prove that if the natural number n is not a perfect square, then $\sqrt{n}$ is irrational.

Solution: Assume that n is a natural number that is not a perfect square i.e., $\forall m\in\mathbb{N}: n\neq m^2$. We will prove that $\sqrt{n}\notin\mathbb{Q}$

- Reasoning by absurd (contraduction). Suppose this fails. that is $\sqrt{n}=\frac{p}{q}$ for some coprime integers p and q ($\gcd(p,q)=1$)

thus: $\sqrt{n} = \frac{p}{q} \Leftrightarrow p^2 = nq^2$

$\Leftrightarrow q^2 \setminus p^2 \qquad (q^2 \text{ divise } p^2)$

$\Rightarrow q^2 = 1 \qquad (\gcd(p,q) = 1)$

$\Rightarrow p^2 = n$ and this contradict the assumption

n is not a perfect square, hence $\sqrt{n} \notin \mathbb{Q}$.

(b) Prove that: if $r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$, then $r + x \notin \mathbb{Q}$

Solution:

We use contraduction; suppose that $r + x \in \mathbb{Q}$.

from $r \in \mathbb{Q}$, we have $-r \in \mathbb{Q}$, thus $(r+x)+(-r) \in \mathbb{Q}$; that is $x \in \mathbb{Q}$

and this contradict the assumption $x \notin \mathbb{Q}$, therefore $r + x \notin \mathbb{Q}$

(c) Prove that: If $r \in \mathbb{Q}^*$ and $x \notin \mathbb{Q}$, then $r + x \notin \mathbb{Q}$.

Solution: Applying the same methods.

(d) Explain why the number $\sqrt{15} + \sqrt{12} \notin \mathbb{Q}$?

Solution:

Reasoning by absurd; assume that $\sqrt{15} + \sqrt{12} \in \mathbb{Q}$, then $(\sqrt{15} + \sqrt{12})^2 \in \mathbb{Q}$

that is $27 + 2\sqrt{180} \in \mathbb{Q}$ so $\sqrt{180} \in \mathbb{Q}$. But 180 is not a perfect square, which means that $\sqrt{180} \notin \mathbb{Q}$, this gives contraduction.

therefore $\sqrt{15} + \sqrt{12} \notin \mathbb{Q}$.

Exercise 4:

Part ① Let E and F be two non-empty and bounded sets.

ⓐ Prove that: $(E \subseteq F) \Rightarrow (\mathrm{Inf}(F) \leq \mathrm{Inf}(E) \leq \mathrm{Sup}(E) \leq \mathrm{Sup}(F))$

Solution:

Assume that $E \subseteq F$, from the fact $\forall x \in F : x \geq \mathrm{Inf}(F)$, we get:

$\forall x \in E : x \geq \mathrm{Inf}(F)$, that is $\mathrm{Inf}(F)$ is a lower bound for E

and since $\mathrm{Inf}(E)$ is the greatest lower bound for E, we obtain:

$\mathrm{Inf}(F) \leq \mathrm{Inf}(E)$.

ⓑ Prove that: $\mathrm{Sup}(E \cup F) = \mathrm{Max}\{\mathrm{Sup}(E), \mathrm{Sup}(F)\}$.

Solution:

We have: $x \in E \cup F \iff x \in E$ or $x \in F$

then $x \leq \mathrm{Sup}(E)$ or $x \leq \mathrm{Sup}(F)$

therefore $\forall x \in E \cup F : \quad x \leq \mathrm{Max}\{\mathrm{Sup}(E), \mathrm{Sup}(F)\}$

so, $\quad \mathrm{Sup}(E \cup F) \leq \mathrm{Max}\{\mathrm{Sup}(E), \mathrm{Sup}(F)\} \quad \cdots (1)$

on the other hand:

$E \subseteq E \cup F \Rightarrow \mathrm{Sup}(E) \leq \mathrm{Sup}(E \cup F)$

$F \subseteq E \cup F \Rightarrow \mathrm{Sup}(F) \leq \mathrm{Sup}(E \cup F)$

thus $\mathrm{Max}\{\mathrm{Sup}(E), \mathrm{Sup}(F)\} \leq \mathrm{Sup}(E \cup F) \quad \cdots (1)$

from the inequalities (1) and (2), we get:

$$\mathrm{Sup}(E \cup F) = \mathrm{Max}\{\mathrm{Sup}(E), \mathrm{Sup}(F)\}.$$

Part ② We set: $E - F = \{x - y \mid x \in E \text{ and } y \in F\}$, $-F = \{-x \mid x \in F\}$

ⓐ Prove that $\mathrm{Sup}(E-F) = \mathrm{Sup}(E) - \mathrm{Inf}(F)$.

Solution:

We have $\mathrm{Sup}(E) = M \iff \begin{cases} \forall x \in E : x \leq M \text{ and} & \cdots (1) \\ \forall \varepsilon > 0, \exists a \in E : a \geq M - \frac{\varepsilon}{2} & \cdots (2) \end{cases}$

also,

$$\text{Inf}(F) = m \iff \begin{cases} \forall y \in F: & y \geq m \quad \cdots\cdots (3) \\ \forall \varepsilon > 0, \exists b \in F: & b \leq m + \frac{\varepsilon}{2} \quad \cdots (4) \end{cases}$$

$$\iff \begin{cases} \forall y \in F: & -y \leq -m \quad \cdots\cdots (5) \\ \forall \varepsilon > 0, \exists b \in F & -b \geq -m - \frac{\varepsilon}{2} \quad \cdots (6) \end{cases}$$

By adding the inequalities (1) and (5) as well as the inequalities (2) and (6),

we get: $\begin{cases} \forall x \in E, \forall y \in F: & x - y \leq M - m \\ \forall \varepsilon > 0, \exists a \in E, \exists b \in F: & a - b \geq M - m - \varepsilon \end{cases}$

thus $\text{Sup}(E-F) = M - m = \text{Sup}(E) - \text{Inf}(F)$.

(b) Similarly, we can prove that: $\text{Inf}(E-F) = \text{Inf}(E) - \text{Sup}(F)$,

(c) Prove that: $\text{Sup}(-F) = -\text{Inf}(F)$.

Solution:

Put $E = \{0\}$ in the relation: $\text{Sup}(E-F) = \text{Sup}(E) - \text{Inf}(F)$, we get:

$\text{Sup}(-F) = -\text{Inf}(F)$. (because $\{0\} - F = -F$)

Part (3) Let $E \subseteq \mathbb{R}_+^*$ and $1/E = \{\frac{1}{x} \mid x \in E\}$

(a) Prove that: $\text{Inf}(1/E) = \dfrac{1}{\text{Sup}(E)}$

Solution:

We have: $\forall x \in E: x \leq \text{Sup}(E)$, then $\frac{1}{x} \geq \frac{1}{\text{Sup}(E)}$, it follows:

$\forall (\frac{1}{x}) \in 1/E: \frac{1}{x} \geq \frac{1}{\text{Sup}(E)}$, and since $\text{Inf}(1/E)$ is the greatest lower bound for $1/E$, we obtain $\text{Inf}(1/E) \geq \dfrac{1}{\text{Sup}(E)} \cdots (1)$

On the other hand, we have: $\forall (\frac{1}{x}) \in 1/E: \frac{1}{x} \geq \text{Inf}(1/E)$ which means $\forall x \in E: x \leq \dfrac{1}{\text{Inf}(1/E)}$ and since $\text{Sup}(E)$ is the least upper bound for E

we get $\text{Sup}(E) \leq \dfrac{1}{\text{Inf}(1/E)} \cdots (2)$. from (1) and (2) we get: $\text{Sup}(E) = \dfrac{1}{\text{Inf}(1/E)}$.