TD N°01

Exercise N°01

① Prove that: $\forall x, y \in \mathbb{R} \quad ||x| - |y|| \leq |x+y|$?

Let $x, y \in \mathbb{R}$, we have

$$|x| = |x+y-y| \leq |x+y| + |-y| \Rightarrow |x| - |y| \leq |x+y|$$

$$|y| = |y+x-x| \leq |x+y| + |-x| \Rightarrow |y| - |x| \leq |x+y|$$

thus $$-|x+y| \leq |x| - |y| \leq |x+y|$$

So, $$||x| - |y|| \leq |x+y| \quad \text{for all } x, y \in \mathbb{R}.$$

② Prove that: $\forall x, y \in \mathbb{R}, \forall \varepsilon > 0 : xy \leq \frac{x^2}{2\varepsilon} + \frac{\varepsilon}{2} y^2$?

Let $x, y \in \mathbb{R}$ and $\varepsilon > 0$

we have $(a-b)^2 \geq 0$ for all $a, b \in \mathbb{R}$

So, $a^2 + b^2 \geq 2ab$ for all $a, b \in \mathbb{R}$.

by Putting $a = \frac{x}{\sqrt{\varepsilon 2}}$, $b = \sqrt{\frac{\varepsilon}{2}}$, we get:

$$xy \leq \frac{x^2}{2\varepsilon} + \frac{\varepsilon}{2} y^2.$$

③ Prove that: $(\forall \varepsilon > 0 : |x| < \varepsilon) \Rightarrow (x = 0)$?

We use the contrapositive proof, which means that; it is sufficient to prove that:

$$(x \neq 0) \Rightarrow (\exists \varepsilon > 0 : |x| \geq \varepsilon) ?$$

so, to prove the statement $(\exists \varepsilon > 0 : |x| \geq \varepsilon)$ is true for $x \neq 0$, it suffices to choose $\varepsilon = \frac{|x|}{2}$.

(4) Prove that: $\forall x, y \in \mathbb{R}: |x|+|y| \leq |x+y|+|x-y|$?

We have $|a+b| \leq |a|+|b|$, $|a-b| \leq |a|+|b|$, for all $a, b \in \mathbb{R}$.

then by summation, we obtain the following:

$$|a+b|+|a-b| \leq 2|a|+2|b| \quad \ldots \text{(1)}$$

Now, by substituting $a=\frac{x+y}{2}$, $b=\frac{x-y}{2}$ into the last inequality (1),

we get: $|x|+|y| \leq |x+y|+|x-y|$ as the desired result.

(5) Prove that: $\left(|x+y|=|x|+|y|\right) \Leftrightarrow (xy \geq 0)$

Let x, y be two real numbers, we have:

$$\begin{aligned}
|x+y|=|x|+|y| \quad &\text{iff} \quad |x+y|^2 = (|x|+|y|)^2 \\
&\text{iff} \quad x^2+y^2+2xy = x^2+y^2+2|xy| \\
&\text{iff} \quad xy=|xy| \\
&\text{iff} \quad xy \geq 0.
\end{aligned}$$

Hence, the desired results.

Exercise 2

Part (1)

(a) Prove that: $\forall \varepsilon>0, \exists n \in \mathbb{N}^*: 0<\frac{1}{n}<\varepsilon$

By setting $x=\varepsilon$, $y=1$ into the Archimedes' axiom:

$$\forall x>0, \forall y \in \mathbb{R}, \exists n \in \mathbb{N}^*: nx>y$$

We obtain:

$$\forall \varepsilon>0, \exists n \in \mathbb{N}^*: n\varepsilon>1$$

which yields:

$$\forall \varepsilon>0, \exists n \in \mathbb{N}^*: 0<\frac{1}{n}<\varepsilon$$

Hence, the required inequality.

(b) Prove that: $\forall x, y \in \mathbb{R}: (x<y) \Rightarrow (E(x) \leq E(y))$.

Let x, y be two real numbers.

If we have $x<y$, then $E(x) \leq x < y$, and since $E(y)$ is the largest integer less than or equal to y, we get: $E(x) \leq E(y)$.

(c)* Prove that: $\forall x, y \in \mathbb{R}: E(x)+E(y) \leq E(x+y) \leq E(x)+E(y)+1$.

(d) Prove that: $\forall x \in \mathbb{R}: -1 \leq E(x)+E(-x) \leq 0$.

Let x is a real number, we substitute $y=-x$ into the following inequality:

$$\forall x, y \in \mathbb{R}: E(x)+E(y) \leq E(x+y) \leq E(x)+E(y)+1$$

we get:

$$\forall x \in \mathbb{R}: \quad E(x)+E(-x) \leq E(0) \leq E(x)+E(-x)+1.$$

therefore:
$$\begin{cases} E(x)+E(-x) \leq 0; \\ E(x)+E(-x) \geq -1. \end{cases}$$

(e) Prove that: $\forall x, y \in \mathbb{R}; \ Max\{x,y\} = \dfrac{x+y+|x-y|}{2}, \ Min\{x,y\} = \dfrac{x+y-|x-y|}{2}$.

Let $x, y \in \mathbb{R}$, we have:

$$x+y = Max\{x,y\} + Min\{x,y\} \quad \dots (1)$$

$$|x-y| = Max\{x,y\} - Min\{x,y\} \quad \dots (2)$$

by summing and subtracting the last two equalities (1) and (2), we get:

$$Max\{x,y\} = \frac{x+y+|x-y|}{2} \quad \text{and}$$

$$Min\{x,y\} = \frac{x+y-|x-y|}{2} \quad \text{respectively.}$$

Exercise 2 :

Part 2 : Determine, if possible the Maximum, minimum, supremum and Infimum for each set from the following :

ⓐ $A = \left\{ \frac{2n+1}{n} \;/\; n \in \mathbb{N}^* \right\}$ ⓑ $B = \left\{ \frac{1}{1+x^2} \;/\; x \in \mathbb{R} \right\}$.

Solution:

ⓐ first of all the set A is not empty (because $3 = \frac{2(1)+1}{1} \in A$).

We must now show whether the set A is bounded from below and above or not :

We have : $\forall n \in \mathbb{N}^* : \quad \frac{2n+1}{n} = 2 + \frac{1}{n}$

then : $2 < 2 + \frac{1}{n} \leq 3$ ... ①

We can only say that 3 is an upper bound for the set A, but in this case, since $3 \in A$, then $Max(A) = 3$, and therefore $Sup(A) = 3$, and we don't need to use the characteristic property of sup.

Now, from the inequality ①, we deduce that A is bounded from below by 2.

to prove that $inf(A) = 2$ by Archimedes' axiom, we use :

$$\forall \varepsilon > 0, \; \exists n \in \mathbb{N}^* : \; \varepsilon n > 1$$

So : $\forall \varepsilon > 0, \; \exists n \in \mathbb{N}^* : \; 0 < \frac{1}{n} < \varepsilon$

then : $\forall \varepsilon > 0, \; \exists n \in \mathbb{N}^* \quad \frac{1}{n} + 2 < 2 + \varepsilon$

thus : $\forall \varepsilon > 0, \; \exists a = \frac{1}{n} + 2 \in A : \; 2 < a < 2 + \varepsilon$.

therefore $Inf(A) = 2$.

Since $2 \notin A$, we conclude that the $Min(A)$ does not exist,