University of Oum El Bouaghi — Academic year: 2025/2026

Introduction to topology — License 2 - Mathematics

Sheet of exercises N°1

Exercise 1 *Let $X =]0, +\infty[$. For $x, y \in X$, we set*

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|.$$

1. *Show that d is a distance on X.*
2. *Show that, in the metric space (X, d), $\mathbb{N}^*$ is a bounded subset and $\left\{\frac{1}{n}\ ;\ n \in \mathbb{N}^*\right\}$ is an unbounded subset.*
3. *Is the metric space (X, d) complete? (Indication. Consider the real sequence $x_n = n$).*

Exercise 2 (Construction of distances) *We call a gauge a map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ which is non-decreasing, only vanishing at 0 and sub-additive (i.e. $\varphi(s+t) \le \varphi(s)+\varphi(t)$, $\forall s, t \in \mathbb{R}_+$).*

1. *Let (X, d) be a metric space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a gauge. Show that $d' = \varphi \circ d$ is a distance on X.*
2. *Application: Deduce that*

$$d_1 = \frac{d}{1+d}, \qquad d_2 = \min\{1, d\}, \qquad d_3 = \ln(1+d), \qquad \text{and} \qquad d_4 = d^\alpha \text{ with } 0 < \alpha < 1$$

   are distances on X.
3. *Show that d and $d_1 = \frac{d}{1+d}$ are topologically equivalent.*
4. *Show that d and $d' = \varphi \circ d$ are metrically equivalent if there exists a real constant $C > 0$ such that*

$$C^{-1}t \le \varphi(t) \le Ct, \qquad \forall t \in \mathbb{R}^+.$$

Exercise 3 *Show that in a metric space every finite subset is closed.*

Exercise 4 *We equip $\mathbb{R}$ with the usual distance.*

1. *Describe the interior and the closure of the set $A = \left\{\frac{1}{n}\ ;\ n \in \mathbb{N}^*\right\}$.*
2. *Specify the accumulation points and the isolated points of A.*

Exercise 5 *In $\mathbb{R}$ equipped with the usual distance, consider the part*

$$A =]-\infty, -1[\cup \left\{0, \frac{\pi}{4}, \sqrt{3}\right\} \cup \left\{3 - \frac{1}{n}\ ;\ n \in \mathbb{N}^*\right\}.$$

1. *Determine $\mathring{A}$ and $\overline{A}$.*

2. *Determine the accumulation points and the isolated points of A.*

Exercise 6 *Let (X, d) be a metric space and let A and B be two non-empty subsets of X. Show that:*

1. $\forall x, y \in X : \quad |d(x, A) - d(y, A)| \leq d(x, y)$,

2. $\forall x \in X : \quad x \in \overline{A} \iff d(x, A) = 0$,

3. $\operatorname{diam}(\overline{A}) = \operatorname{diam}(A)$.

Exercise 7 *Show that in a metric space,*

1. *Every convergent sequence is a Cauchy sequence. The converse is false.*

2. *Every Cauchy sequence is bounded. The converse is false.*

3. *Every subsequence of a Cauchy sequence is a Cauchy sequence.*

4. *Every Cauchy sequence with a convergent subsequence is convergent.*

Exercise 8 *Let d and d' be two metrically equivalent distances on a set X. Show that:*

1. *(X, d) and (X, d') have the same bounded parts.*

2. *(X, d) and (X, d') have the same convergent sequences and the same Cauchy sequences.*

3. *(X, d) is complete if and only if (X, d') is complete.*

Exercise 9 *Let d and d' be two distances on X. Show that*

1. *d is finer than d' if and only if the identity map $\mathsf{id} : (X, d) \to (X, d')$ is continuous.*

2. *d and d' are topologically equivalent if and only if the identity map $\mathsf{id} : X \to X$ is a homeomorphism from (X, d) onto (X, d').*

Exercise 10 *Let (X, d) be a metric space.*

1. *Given a non-empty subset A of X, show that the map $d(\cdot, A) : x \mapsto d(x, A)$ from X to $\mathbb{R}$ is Lipschitzian.*

2. *We equip the product set $X \times X$ with the distance*

$$D_\infty((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1), d(x_2, y_2)\}$$

Show that the map $d : X \times X \to \mathbb{R}$ is Lipschitzian.