University of Oum El Bouaghi | Academic year: 2025/2026
Introduction to topology | License 2 - Mathematics

Sheet of exercises N°0

For a set X, we denote $\mathcal{P}(X)$ the set of its subsets, and for $A \in \mathcal{P}(X)$ we denote $A^\complement$ its complement in X.

Exercice 1 *Describe* $\mathcal{P}(X)$ *and* $\mathcal{P}(\mathcal{P}(X))$ *for*

(a) $X = \varnothing$; *(b)* $X = \{1\}$; *(c)* $X = \{1, 2\}$.

Exercice 2 *For all subsets* A *and* B *of a set* X*, show the following equivalences:*

1. $A \subset B \iff A \cap B = A$
2. $A \subset B \iff A \cup B = B$
3. $A \subset B \iff A \setminus B = \varnothing$
4. $A \cap B = \varnothing \iff A \subset B^\complement$
5. $A \setminus B = A \iff A \cap B = \varnothing$
6. $A \subset B \iff A^\complement \supset B^\complement$.

Exercice 3 *Let* X *be a set and let* $(A_i)_{i\in I}$ *be any family in* $\mathcal{P}(X)$*. For* $I = \varnothing$*, we agree to set* $\bigcup_{i\in I} A_i = \varnothing$ *and* $\bigcap_{i\in I} A_i = X$*. Establish the relations:*

1. $\left(\bigcup_{i\in I} A_i\right)^\complement = \bigcap_{i\in I} A_i^\complement$
2. $\left(\bigcap_{i\in I} A_i\right)^\complement = \bigcup_{i\in I} A_i^\complement$
3. $A \cap \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(A \cap A_i)$
4. $A \cup \left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I}(A \cup A_i)$

Exercice 4

1. *Consider the map* $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x + 1$*. Determine the (direct) image of each of the following sets:*

$$A = \{1, 2, 4\}, \quad B = \{x \in \mathbb{Z} \; ; \; x \geq -1\}, \quad C = \mathbb{N}.$$

2. *Consider the map* $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2 - 1$. *Determine the preimage of each of the following sets:*

$$A = \{2, 3, 7\}, \quad B = \mathbb{N}, \quad C = [-1, +\infty[\,.$$

3. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be the map defined by:*

$$f(x) = \begin{cases} x + 1, & x \geq 0, \\ 1, & x < 0. \end{cases}$$

   Determine the direct image of $\mathbb{R}$ *and then the preimage of the interval* $[1, 2]$.

Exercice 5 *Let* X *and* Y *be two sets and* f *a map from* X *to* Y. *Show that for any subset* A *of* X *and any subset* B *of* Y, *we have*

1. $f\left(f^{-1}(B)\right) \subset B$ *and* $A \subset f^{-1}\left(f(A)\right)$,
2. $f^{-1}\left(B^{\complement}\right) = \left(f^{-1}(B)\right)^{\complement}$.

Exercice 6 *Let* X *and* Y *be two sets and* f *a map from* X *to* Y.

1. *For any families* $(A_i)_{i \in I}$ *of* $\mathcal{P}(X)$ *and* $(B_i)_{i \in I}$ *of* $\mathcal{P}(Y)$, *show that we have:*

   i) $f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$

   ii) $f\left(\bigcap_{i \in I} A_i\right) \subset \bigcap_{i \in I} f(A_i)$

   iii) $f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$

   iv) $f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)$

2. *Give an example of strict inclusion for the second relation.*

3. *Show that equality holds in the second relation if and only if* f *is injective.*

Exercice 7 *Let the map* $f : \mathbb{R} \to \mathbb{R},\ x \mapsto |x| + 2x$.

1. *Show that* f *is bijective.*

2. *Determine the inverse map* f^{-1}.