

Tutorial n°3: Kinematics of material point

(Without change of reference)

Exercise 1

The rectilinear movement of mobile M along the $x'Ox$ axis is defined by: $x(t) = 2t^3 + 5t^2 + 5$ (m)

- 1- Determine the expression of the velocity and acceleration of point M as a function of t .
- 2- Calculate its position, velocity and acceleration at $t_1 = 2s$ and $t_2 = 3s$.
- 3- Calculate the average velocity and the average acceleration between t_1 and t_2 .

Exercise 2

The velocity of a particle moving in the (Oxy) plane at any time t is given by:
$$\begin{cases} V_x(t) = 4t^3 + 4t \\ V_y(t) = 4t \end{cases}$$

At time $t = 0s$ the mobile is at position (1, 2).

- 1- Find the trajectory equation of the movement $y = f(x)$ and determine its nature.
- 2- Find the components of the acceleration vector $\vec{\gamma}$.
- 3- Calculate the scalar product $\vec{V} \cdot \vec{\gamma}$ and vector product $\vec{V} \wedge \vec{\gamma}$.

Exercise 3

A mobile M moves on a curvilinear trajectory of radius $R = 10$ m, where: $S(t) = 2t^3 - 5t^2 - 2$

- 1- Find velocity and acceleration vectors of M in the intrinsic coordinate system.
- 2- Calculate the magnitude of the velocity and acceleration vectors at time $t = 2$ s.

Exercise 4

In the Cartesian coordinate system (Oxy) provided with a base (\vec{i}, \vec{j}) , we give:
$$\begin{cases} x = 1 + 2 \cos \beta \\ y = 2 \sin \beta \end{cases}$$

The relation between the angle β and the time t is given by: $\beta = 2t + 2$.

- 1- Find the trajectory equation of mobile and deduce the nature of movement.
- 2- Give the expression of the position vector \overline{OM} .
- 3- Find the derivative of angle β with respect to time.
- 4- Determine the components of the velocity vector \vec{V} and its magnitude $\|\vec{V}\|$.
- 5- Determine the components of the acceleration vector $\vec{\gamma}$ and its magnitude $\|\vec{\gamma}\|$.

Exercise 5

The curvilinear movement of a particle M is identified in polar system by the following equations:

$$r(t) = r_0 e^{-\frac{t}{a}}; \quad \theta(t) = \frac{t}{a}$$

r_0 and a are positive constants.

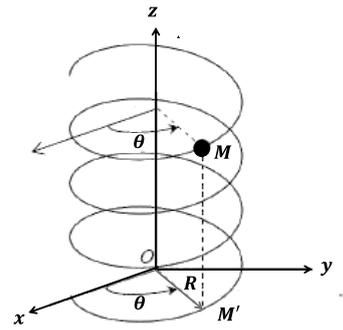
- 1- Give the expression of the position vector.
- 2- Find the expressions of the velocity and acceleration vectors as well as their magnitudes.
- 3- Calculate the normal γ_N and tangential γ_T components of the acceleration vector in the intrinsic coordinate system, then deduce the radius of curvature R_c of the trajectory.

Exercise 6

A material point M describes a circular helix with Oz axis, its movement is given in Cartesian by:

$$\begin{cases} x = R \cos \theta \\ y = R \sin \theta \\ z = h \theta \end{cases}$$

R is the radius of the helix and θ is the angle that OM' makes with the Ox axis ($\theta = \omega t$). R, ω and h are positive constants.

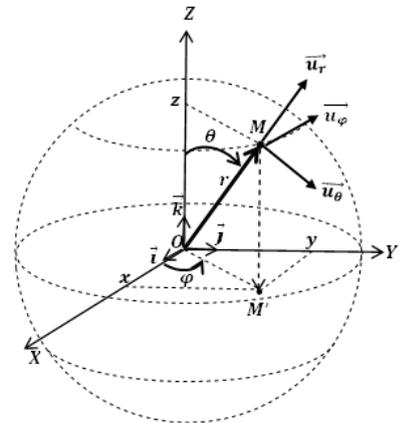


- 1- Determine in cylindrical coordinate system the expression of the position, velocity and acceleration vectors.
- 2- Is the magnitude of velocity constant? What is the nature of this movement?

Exercise 7

In the spherical coordinate system $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi)$, the movement of a mobile M is defined by the elements (r, θ, φ) , as shown in **the figure opposite**.

- 1- Determine the relation between (x, y, z) and (r, θ, φ) .
- 2- Give the position vector \vec{r} .
- 3- Determine the expression of the unit vectors $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi)$ as a function of the unit vectors $(\vec{i}, \vec{j}, \vec{k})$
- 4- Calculate the time derivative of: $\frac{d\vec{u}_r}{dt}$, $\frac{d\vec{u}_\theta}{dt}$ and $\frac{d\vec{u}_\varphi}{dt}$.
- 5- Determine the velocity and acceleration vectors.



Exercise 8

In the spherical coordinate system $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi)$. A point M moves on the surface of a sphere of radius R . Its spherical coordinates are:

$$r = R; \quad \theta = \frac{\pi}{6}; \quad \varphi = \omega t^2$$

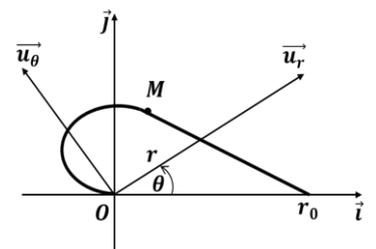
- 1- Find the velocity and acceleration vectors of M , and calculate its magnitudes.
- 2- Deduce the normal acceleration γ_N .

Exercise 9

Consider a material point M moving in a plane (O, \vec{i}, \vec{j}) along the trajectory shown in the **figure below**. The equation of this trajectory is given in polar coordinates system by:

$$r = \frac{r_0}{2} (1 + \cos \theta); \quad r_0: c^t$$

- 1- Determine the velocity vector \vec{V} in the polar coordinates system. Deduce its magnitude.
- 2- Determine the tangential unit vector \vec{u}_T in the Frenet basis (intrinsic coordinate system). Deduce that this vector forms an angle $\theta/2$ with the polar unit vector \vec{u}_r .
- 3- Determine the acceleration $\vec{\gamma}$ in the intrinsic coordinate system.
- 4- Determine the radius of curvature R and the normal unit vector \vec{u}_N .
- 5- Determine the curvilinear abscissa S of M , counted from the point corresponding to $\theta = 0$?
- 6- Deduce the total length of the trajectory L .



Exercise-01

We have: $x(t) = 2t^3 + 5t^2 + 5 (m)$

1) **Velocity and acceleration**

$$V(t) = \frac{dx}{dt} = 6t^2 + 10t$$

$$\gamma(t) = \frac{dV}{dt} = 12t + 10$$

2) **Position, velocity and acceleration at t_1 and t_2**

$$t_1 = 2s \begin{cases} x_1 = 41 (m) \\ V_1 = 44 (m \cdot s^{-1}) \\ \gamma_1 = 34 (m \cdot s^{-2}) \end{cases}$$

$$t_2 = 3s \begin{cases} x_2 = 104 (m) \\ V_2 = 84 (m \cdot s^{-1}) \\ \gamma_2 = 46 (m \cdot s^{-2}) \end{cases}$$

3) **Average velocity and average acceleration**

$$V_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = 63 (m \cdot s^{-1})$$

$$\gamma_{avg} = \frac{\Delta V}{\Delta t} = \frac{V_2 - V_1}{t_2 - t_1} = 40 (m \cdot s^{-2})$$

Exercise -02

We have: $\begin{cases} V_x(t) = 4t^3 + 4t \\ V_y(t) = 4t \end{cases}$

1) **Trajectory equation $y = f(x)$**

• **Determining the equations of x and y**

$$V_x = \frac{dx}{dt} \Rightarrow x = \int V_x dt = \int (4t^3 + 4t) dt$$

$$V_y = \frac{dy}{dt} \Rightarrow y = \int V_y dt = \int 4t dt$$

By integrating:

$$x = t^4 + 2t^2 + c_1; (c : integrality constant)$$

$$y = 2t^2 + c_2$$

According to the initial conditions at $t = 0 s$:

$$x(0) = c_1 = 1$$

$$y(0) = c_2 = 2$$

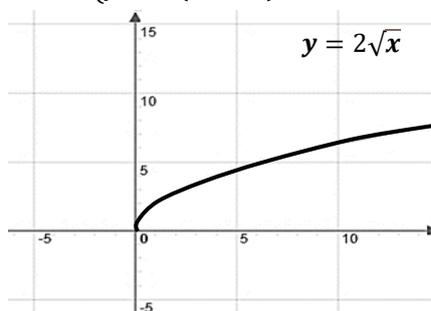
We finally obtain the x and y :

$$x = t^4 + 2t^2 + 1$$

$$y = 2t^2 + 2$$

• **Equation $y = f(x)$**

$$\text{We have: } \begin{cases} x = (t^2 + 1)^2 \\ y = 2(t^2 + 1) \end{cases} \Rightarrow y = 2\sqrt{x}$$



The nature of the trajectory $y = 2\sqrt{x}$ is parabolic

2) **Components of acceleration vector $\vec{\gamma}$**

$$\begin{cases} \gamma_x = \frac{dV_x}{dt} = 12t^2 + 4 \\ \gamma_y = \frac{dV_y}{dt} = 4 \end{cases}$$

3) **Scalar product $\vec{V} \cdot \vec{\gamma}$ and vector product $\vec{V} \wedge \vec{\gamma}$**

The vectors are written: $\begin{cases} \vec{V} = (4t^3 + 4t)\vec{i} + (4t)\vec{j} \\ \vec{\gamma} = (12t^2 + 4)\vec{i} + 4\vec{j} \end{cases}$

$$\triangleright \vec{V} \wedge \vec{\gamma} = -32t^3\vec{k}$$

$$\triangleright \vec{V} \cdot \vec{\gamma} = 16(3t^5 + 4t^3 + 2t)$$

Exercise -03

1) **velocity and acceleration vectors of M**

in the intrinsic coordinate system:

Velocity:

$$\vec{V} = V \vec{u}_T; \quad V = \frac{dS}{dt}$$

$$\vec{V} = (6t^2 - 10t) \vec{u}_T$$

Acceleration:

$$\vec{\gamma} = \gamma_T \vec{u}_T + \gamma_N \vec{u}_N \Rightarrow \begin{cases} \gamma_T = \frac{dV}{dt} \\ \gamma_N = \frac{V^2}{R} \end{cases}$$

$$\vec{\gamma} = (12t - 10) \vec{u}_T + \frac{(6t^2 - 10t)^2}{10} \vec{u}_N$$

2) **Magnitude of the velocity and acceleration vectors**

at time $t = 2 s$

$$V = 4 m \cdot s^{-1}; \quad \gamma = 14.1 m \cdot s^{-2}$$

Exercise -04

1) **Trajectory equation $y = f(x)$**

Using the relationship: $\cos^2 \beta + \sin^2 \beta = 1$

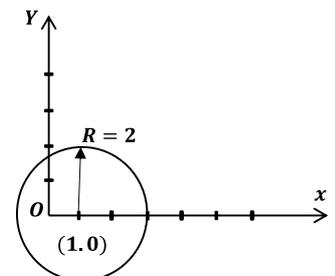
$$\Rightarrow (x - 1)^2 + y^2 = 2^2$$

The trajectory is a **circle** of

radius **$R=2$** and center **$(1,$**

$0)$;

Nature of Mvt: Circular movement



2) **Position vector \vec{OM}**

$$\vec{OM} = x \vec{i} + y \vec{j}$$

$$\Rightarrow \vec{OM} = (2 \cos \beta + 1)\vec{i} + 2 \sin \beta \vec{j}$$

3) **The temporal derivative of the angle β :**

$$\beta = 2t + 2 \Rightarrow \frac{d\beta}{dt} = 2$$

4) **Components and magnitude of the velocity vector \vec{V}**

$$\vec{V} = V_x \vec{i} + V_y \vec{j}$$

$$\begin{cases} V_x = \frac{dx}{dt} = \frac{dx}{d\beta} \cdot \frac{d\beta}{dt} \\ V_y = \frac{dy}{dt} = \frac{dy}{d\beta} \cdot \frac{d\beta}{dt} \end{cases} \Rightarrow \begin{cases} V_x = -4 \sin \beta \\ V_y = +4 \cos \beta \end{cases}$$

$$\Rightarrow \vec{V} = 4(-\sin\beta \vec{i} + \cos\beta \vec{j})$$

$$\|\vec{V}\| = \sqrt{V_x^2 + V_y^2} = 4$$

5) Components and magnitude of the acceleration vector $\vec{\gamma}$

$$\vec{\gamma} = \gamma_x \vec{i} + \gamma_y \vec{j}$$

$$\begin{cases} \gamma_x = \frac{dV_x}{dt} = \frac{dV_x}{d\beta} \cdot \frac{d\beta}{dt} \\ \gamma_y = \frac{dV_y}{dt} = \frac{dV_y}{d\beta} \cdot \frac{d\beta}{dt} \end{cases} \Rightarrow \begin{cases} \gamma_x = -8 \cos \beta \\ \gamma_y = -8 \sin \beta \end{cases}$$

$$\Rightarrow \vec{\gamma} = -8(\cos\beta \vec{i} + \sin\beta \vec{j})$$

$$\|\vec{\gamma}\| = \sqrt{\gamma_x^2 + \gamma_y^2} = 8$$

Exercise-05

We have: $\begin{cases} \vec{r}(t) = r_0 e^{-\frac{t}{a}} \\ \theta(t) = \frac{t}{a} \end{cases}$

1) The expression of the position vector \vec{OM}

$$\vec{OM} = r \vec{u}_r$$

$$\Rightarrow \vec{OM} = (r_0 e^{-\frac{t}{a}}) \vec{u}_r$$

2) The velocity and acceleration vectors

• Velocity vector and its magnitude

We have: $\vec{V} = V_r \vec{u}_r + V_\theta \vec{u}_\theta$

$$\Rightarrow \begin{cases} V_r = \frac{dr}{dt} \\ V_\theta = r \frac{d\theta}{dt} \end{cases} \Rightarrow \begin{cases} V_r = -\frac{r_0}{a} e^{-\frac{t}{a}} \\ V_\theta = \frac{r_0}{a} e^{-\frac{t}{a}} \end{cases}$$

Thus:

$$\vec{V} = \frac{r_0}{a} e^{-\frac{t}{a}} (-\vec{u}_r + \vec{u}_\theta)$$

Magnitude:

$$\|\vec{V}\| = \sqrt{V_r^2 + V_\theta^2} = \frac{\sqrt{2} r_0}{a} e^{-\frac{t}{a}}$$

• Acceleration vector and its magnitude

We have: $\vec{\gamma} = \gamma_r \vec{u}_r + \gamma_\theta \vec{u}_\theta$

$$\Rightarrow \begin{cases} \gamma_r = \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt}\right)^2 \\ \gamma_\theta = 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2} \end{cases} \Rightarrow \begin{cases} \gamma_r = 0 \\ \gamma_\theta = -\frac{2 r_0}{a^2} e^{-\frac{t}{a}} \end{cases}$$

Thus:

$$\vec{\gamma} = -\frac{2 r_0}{a^2} e^{-\frac{t}{a}} \vec{u}_\theta$$

Magnitude:

$$\|\vec{\gamma}\| = \sqrt{\gamma_r^2 + \gamma_\theta^2} = \frac{2 r_0}{a^2} e^{-\frac{t}{a}}$$

3) The normal and tangential components of the acceleration vector γ_N and γ_T

In the intrinsic coordinate system, we have:

$$\vec{\gamma} = \gamma_T \vec{u}_T + \gamma_N \vec{u}_N$$

$$\Rightarrow \begin{cases} \gamma_T = \frac{dV}{dt} \\ \gamma_N = \sqrt{\gamma^2 - \gamma_T^2} \end{cases} \Rightarrow \begin{cases} \gamma_T = -\frac{\sqrt{2} r_0}{a^2} e^{-\frac{t}{a}} \\ \gamma_N = \frac{\sqrt{2} r_0}{a^2} e^{-\frac{t}{a}} \end{cases}$$

The radius of curvature R_c

$$\gamma_T = \frac{V^2}{R_c} \Rightarrow R_c = \frac{V^2}{\gamma_T} \Rightarrow R_c = \sqrt{2} r_0 e^{-\frac{t}{a}}$$

Exercise -06

1) The position, velocity and acceleration vectors

The position vector

We have: $\vec{OM} = \rho \vec{u}_\rho + z \vec{k}$

and: $\rho = \sqrt{x^2 + y^2} = R$

$$\Rightarrow \vec{OM} = R \vec{u}_\rho + h\omega t \vec{k}$$

The velocity vector

$$\vec{V} = \frac{d\vec{OM}}{dt}$$

$$\vec{V} = R \frac{d\vec{u}_\rho}{dt} + \frac{d(h\omega t)}{dt} \vec{k} = R \frac{d\theta}{dt} \frac{d\vec{u}_\rho}{d\theta} + \frac{d(h\omega t)}{dt} \vec{k}$$

$$\Rightarrow \vec{V} = R\omega \vec{u}_\theta + h\omega \vec{k}$$

The acceleration vector

$$\vec{\gamma} = \frac{d\vec{V}}{dt}$$

$$\vec{\gamma} = R\omega \frac{d\vec{u}_\theta}{dt} + 0 \vec{k} = R\omega \frac{d\theta}{dt} \frac{d\vec{u}_\theta}{d\theta}$$

$$\Rightarrow \vec{\gamma} = -R\omega^2 \vec{u}_\rho$$

2) The magnitude of velocity and the type of movement

$$\|\vec{V}\| = \omega \sqrt{R^2 + h^2} = C t$$

Type of movement: Uniform helical movement

Exercise -08

1) velocity \vec{V} and acceleration $\vec{\gamma}$ vectors

In the spherical basis:

Position vector: $\vec{OM} = r \vec{u}_r$

Velocity vector:

$$\vec{V} = \frac{dr}{dt} \vec{u}_r + r \frac{d\theta}{dt} \vec{u}_\theta + r \frac{d\varphi}{dt} \sin \theta \vec{u}_\varphi$$

$$\begin{cases} r = R \Rightarrow \frac{dr}{dt} = 0 \\ \theta = \frac{\pi}{6} \Rightarrow \frac{d\theta}{dt} = 0 \\ \varphi = \omega t^2 \Rightarrow \frac{d\varphi}{dt} = 2\omega t \end{cases}$$

$$\vec{V} = R\omega t \vec{u}_\varphi$$

Acceleration vector:

$$\vec{\gamma} = \frac{d\vec{V}}{dt} = \frac{d(R\omega t)}{dt} \vec{u}_\varphi + R\omega t \frac{d(\vec{u}_\varphi)}{dt}$$

$$\frac{d\vec{u}_\varphi}{dt} = -\frac{d\varphi}{dt} [\sin \theta \vec{u}_r + \cos \theta \vec{u}_\theta]$$

$$\vec{\gamma} = \frac{d\vec{V}}{dt} = -R\omega^2 t^2 \vec{u}_r - \sqrt{3}R\omega^2 t^2 \vec{u}_\theta + R\omega \vec{u}_\varphi$$

2) Magnitudes $\|\vec{V}\|$ and $\|\vec{\gamma}\|$

$$\|\vec{V}\| = R\omega t$$

$$\|\vec{\gamma}\| = \sqrt{4R^2\omega^4 t^4 + R^2\omega^2}$$

3) Normal acceleration γ_N

$$\gamma = \sqrt{\gamma_T^2 + \gamma_N^2} \Rightarrow \gamma_N = \sqrt{\gamma^2 - \gamma_T^2}$$

$$\gamma_T = \frac{dV}{dt} = R\omega$$

$$\gamma_N = 2R\omega^2 t^2$$

Exercise -09

1) Velocity vector \vec{V} in the polar basis

We have: $\vec{OM} = r \vec{u}_r$

$$\vec{OM} = \frac{r_0}{2} (1 + \cos \theta) \vec{u}_r$$

$$\vec{V} = \frac{d\vec{OM}}{dt} = -\frac{r_0}{2} \frac{d\theta}{dt} \sin \theta \vec{u}_r + \frac{r_0}{2} (1 + \cos \theta) \frac{d\theta}{dt} \vec{u}_\theta$$

Knowing that: $\begin{cases} \cos(2\theta) = 2 \cos^2 \theta - 1 \\ \sin(2\theta) = 2 \sin \theta \cos \theta \end{cases}$

$$\vec{V} = r_0 \frac{d\theta}{dt} \cos \left(\frac{\theta}{2} \right) \left[-\sin \left(\frac{\theta}{2} \right) \vec{u}_r + \cos \left(\frac{\theta}{2} \right) \vec{u}_\theta \right]$$

$$\text{magnitude: } V = r_0 \frac{d\theta}{dt} \cos \left(\frac{\theta}{2} \right)$$

2) Tangential unit vector \vec{u}_T

$$\vec{u}_T = \frac{\vec{V}}{V} \Rightarrow$$

$$\vec{u}_T = -\sin \left(\frac{\theta}{2} \right) \vec{u}_r + \cos \left(\frac{\theta}{2} \right) \vec{u}_\theta$$

Angle between \vec{u}_T and \vec{u}_θ is $\theta/2$

$$\vec{u}_T \cdot \vec{u}_\theta = u_T \cdot u_\theta \cos(\vec{u}_T, \vec{u}_\theta) = \cos \left(\frac{\theta}{2} \right)$$

3) Acceleration $\vec{\gamma}$ in the intrinsic basis

$$\vec{V} = V \vec{u}_T$$

$$\vec{V} = r_0 \frac{d\theta}{dt} \cos \left(\frac{\theta}{2} \right) \vec{u}_T$$

$$\vec{\gamma} = \frac{d\vec{V}}{dt} = r_0 \frac{d^2\theta}{dt^2} \cos \left(\frac{\theta}{2} \right) \vec{u}_T - r_0 \left(\frac{d\theta}{dt} \right)^2 \sin \left(\frac{\theta}{2} \right) \vec{u}_T + r_0 \frac{d\theta}{dt} \cos \left(\frac{\theta}{2} \right) \frac{d\vec{u}_T}{dt}$$

$$\frac{d\vec{u}_T}{dt} = \frac{3}{2} \frac{d\theta}{dt} \vec{u}_N$$

$$\vec{\gamma} = \left[r_0 \frac{d^2\theta}{dt^2} \cos \left(\frac{\theta}{2} \right) - \frac{r_0}{2} \left(\frac{d\theta}{dt} \right)^2 \sin \left(\frac{\theta}{2} \right) \right] \vec{u}_T + \left[\frac{3 r_0}{2} \left(\frac{d\theta}{dt} \right)^2 \cos \left(\frac{\theta}{2} \right) \right] \vec{u}_N$$

4) Radius of curvature R and vector \vec{u}_N

We have: $\vec{\gamma} = \frac{v^2}{R}$

$$R = \frac{2 r_0}{3} \cos \left(\frac{\theta}{2} \right)$$

5) Curvilinear coordinate S

$$V = \frac{dS}{dt} = r_0 \frac{d\theta}{dt} \cos \left(\frac{\theta}{2} \right)$$

$$S = r_0 \int_0^\theta \cos \left(\frac{\theta}{2} \right) d\theta$$

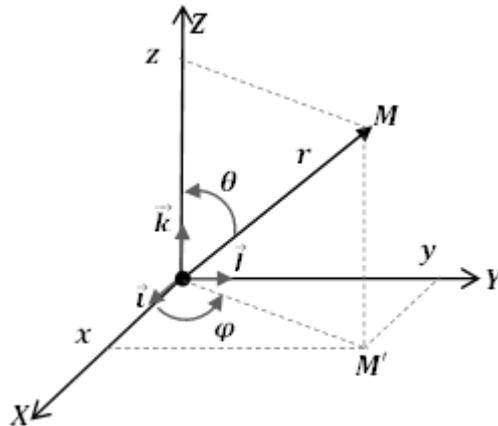
$$S = 2r_0 \sin \left(\frac{\theta}{2} \right)$$

6) Deduce the total length of the trajectory L

$$\theta = \pi; \quad L = 2r_0$$

Exercise 07

1/ The relations between (x, y, z) and (r, θ, φ) are graphically given by:



By projection:

$$\begin{cases} x = OM' \cos \varphi = OM \sin \theta \cos \varphi \\ y = OM' \sin \varphi = OM \sin \theta \sin \varphi \\ z = OM \cos \theta \end{cases}$$

Therefore:

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \quad \text{or} \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \\ \varphi = \arctan\left(\frac{y}{x}\right) \end{cases}$$

2/ Position vector $\vec{r}(t)$

$$\vec{r} = \overrightarrow{OM} = r \vec{u}_r$$

3/ Expression of the unit vectors ($\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi$) as a function of the unit vectors ($\vec{i}, \vec{j}, \vec{k}$)

The unit vector \vec{u}_r :

In the Cartesian reference frame, we have:

$$\begin{aligned} \vec{r} &= x \vec{i} + y \vec{j} + z \vec{k} \\ \Rightarrow \vec{r} &= r \sin \theta \cos \varphi \vec{i} + r \sin \theta \sin \varphi \vec{j} + r \cos \theta \vec{k} \\ \vec{u}_r &= \frac{\vec{r}}{r} = \sin \theta \cos \varphi \vec{i} + \sin \theta \sin \varphi \vec{j} + \cos \theta \vec{k} \end{aligned}$$

The unit vector \vec{u}_θ :

\vec{u}_θ is perpendicular to $\vec{u}_r \Rightarrow \theta \text{ of } \vec{u}_r + \frac{\pi}{2}$:

$$\vec{u}_\theta = \sin\left(\theta + \frac{\pi}{2}\right) \cos \varphi \vec{i} + \sin\left(\theta + \frac{\pi}{2}\right) \sin \varphi \vec{j} + \cos\left(\theta + \frac{\pi}{2}\right) \vec{k}$$

$$\vec{u}_\theta = \cos \theta \cos \varphi \vec{i} + \cos \theta \sin \varphi \vec{j} - \sin \theta \vec{k}$$

The unit vector \vec{u}_φ :

$$\begin{aligned}\vec{u}_\varphi &= \vec{u}_r \wedge \vec{u}_\theta \\ \vec{u}_\varphi &= -\sin \varphi \vec{i} + \cos \varphi \vec{j}\end{aligned}$$

Therefore:

$$\begin{cases} \vec{u}_r = \sin \theta \cos \varphi \vec{i} + \sin \theta \sin \varphi \vec{j} + \cos \theta \vec{k} \\ \vec{u}_\theta = \cos \theta \cos \varphi \vec{i} + \cos \theta \sin \varphi \vec{j} - \sin \theta \vec{k} \\ \vec{u}_\varphi = -\sin \varphi \vec{i} + \cos \varphi \vec{j} \end{cases}$$

4/ Time derivative of: $\frac{d\vec{u}_r}{dt}$, $\frac{d\vec{u}_\theta}{dt}$ and $\frac{d\vec{u}_\varphi}{dt}$.

The radial unit vector \vec{u}_r depend on two variables θ and φ . So, its temporal derivative is given by:

$$\frac{d\vec{u}_r}{dt} = \frac{d\vec{u}_r}{d\theta} \frac{d\theta}{dt} + \frac{d\vec{u}_r}{d\varphi} \frac{d\varphi}{dt} \quad \text{where,} \quad \begin{cases} \frac{d\vec{u}_r}{d\theta} = \vec{u}_\theta \\ \frac{d\vec{u}_r}{d\varphi} = \sin \theta \vec{u}_\varphi \end{cases}$$

$$\frac{d\vec{u}_r}{dt} = \frac{d\theta}{dt} \vec{u}_\theta + \sin \theta \frac{d\varphi}{dt} \vec{u}_\varphi$$

$$\frac{d\vec{u}_\theta}{dt} = -\frac{d\theta}{dt} \vec{u}_r + \cos \theta \frac{d\varphi}{dt} \vec{u}_\varphi$$

$$\frac{d\vec{u}_\varphi}{dt} = \frac{d}{dt} (\vec{u}_r \wedge \vec{u}_\theta) = -\sin \theta \frac{d\varphi}{dt} \vec{u}_r - \cos \theta \frac{d\varphi}{dt} \vec{u}_\theta$$

5/ Velocity vector

$$\begin{aligned}\vec{V} &= \frac{d\vec{OM}}{dt} = \frac{d\vec{r}}{dt} = \frac{d}{dt} (r \vec{u}_r) = \frac{dr}{dt} \vec{u}_r + r \frac{d\vec{u}_r}{dt} \\ \vec{V} &= \frac{dr}{dt} \vec{u}_r + r \frac{d\theta}{dt} \vec{u}_\theta + r \sin \theta \frac{d\varphi}{dt} \vec{u}_\varphi\end{aligned}$$

6/ Acceleration vector

$$\vec{\gamma} = \frac{d\vec{v}}{dt} = \frac{d}{dt} \left[\frac{dr}{dt} \vec{u}_r + r \frac{d\theta}{dt} \vec{u}_\theta + r \sin \theta \frac{d\varphi}{dt} \vec{u}_\varphi \right]$$

$$\vec{\gamma} = \gamma_r \vec{u}_r + \gamma_\theta \vec{u}_\theta + \gamma_\varphi \vec{u}_\varphi$$

$$\begin{cases} \gamma_r = \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 - r \sin^2 \theta \left(\frac{d\varphi}{dt} \right)^2 \\ \gamma_\theta = 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2} - r \cos \theta \sin \theta \left(\frac{d\varphi}{dt} \right)^2 \\ \gamma_\varphi = 2r \cos \theta \frac{d\theta}{dt} \frac{d\varphi}{dt} + 2 \frac{dr}{dt} \frac{d\varphi}{dt} \sin \theta + r \sin \theta \frac{d^2 \varphi}{dt^2} \end{cases}$$