Tutorial n°2: Vector calculus

Exercise 1

In the Cartesian system $(Oxyz)$ provided with an orthonormal basis $(\vec{i},\ \vec{j},\ \vec{k})$, we consider the two following vectors:

$$\overrightarrow{V_1} = 3\,\vec{i} + 6\,\vec{j} + 3\,\vec{k} \quad \text{and} \quad \overrightarrow{V_2} = -\,\vec{i} + \vec{j} + 2\,\vec{k}$$

1- Calculate the sum vector $\vec{S} = \overrightarrow{V_1} + \overrightarrow{V_2}$ and the difference vector $\vec{D} = \overrightarrow{V_1} - \overrightarrow{V_2}$.

2- Represent graphicly the vectors $\overrightarrow{V_1}$, $\overrightarrow{V_2}$, $\vec{S}$ and $\vec{D}$ in 3D-space $(Oxyz)$. **(Turn the page)**

3- Calculate the scalar product $\overrightarrow{V_1} \bullet \overrightarrow{V_2}$ and the vector product $\overrightarrow{V_1} \wedge \overrightarrow{V_2}$.

4- Deduce the angle α included between $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$.

5- Give the projection of the vectors: $\overrightarrow{V_1}/\overrightarrow{V_2}$ and $\overrightarrow{V_2}/\overrightarrow{V_1}$.

Exercise 2

In an orthonormal basis $(\vec{i},\ \vec{j})$, we consider three points: $A(1,0,2)$, $B(-2,1,4)$ and $C(0,3,5)$

1- Determine the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$.

2- Calculate the magnitude of vectors $\|\overrightarrow{AB}\|$, $\|\overrightarrow{AC}\|$, and $\|\overrightarrow{BC}\|$.

3- Deduce the unit vectors $\overrightarrow{u_{AB}}$, $\overrightarrow{u_{AC}}$, and $\overrightarrow{u_{BC}}$.

4- Calculate the vector product $\overrightarrow{AB} \wedge \overrightarrow{AC}$, and deduce the area of triangle ABC.

Exercise 3

In an orthonormal basis $(\vec{i},\ \vec{j},\ \vec{k})$, we give the following points: $A\ (2,-3,1)$; $B\ (5,5,-4)$; $C\ (0,0,4)$

1- Determine the vector: $\vec{S} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$.

2- Calculate the directional cosines of the vector $\vec{S}$.

3- Give the projection of the vector $\vec{S}$ on the axes: Ox, Oy, Oz, and on the straight line (Δ): y = x

4- Calculate the vector product $\overrightarrow{OB} \wedge \overrightarrow{OC}$ and deduce the surface area of triangle OBC.

5- Calculate the mixed product $\overrightarrow{OA} \bullet (\overrightarrow{OB} \wedge \overrightarrow{OC})$ and deduce the volume of the parallelepiped formed by these vectors.

6- Calculate the moment of the vector $\vec{S}$ about a point $I\ (1,0,4)$ **and** about a straight line (IC).

Exercise 4

In an orthonormal basis $(\vec{i},\ \vec{j},\ \vec{k})$, we consider the following forces**:**

$$\overrightarrow{F_1} = 3\vec{i} + \vec{j} + 4\vec{k}; \qquad \overrightarrow{F_2} = \frac{1}{2}\vec{i} - \vec{j} + \frac{a}{2}\vec{k}; \qquad \overrightarrow{F_3} = a\vec{i} + 2\vec{j} - a\vec{k} \qquad (N)$$

Where a is a constant, these forces act at the points $A(1,-2,a)$, $B(6,2,8)$ and $C(1,0,-1)$.

1- Determine the vectors: $\overrightarrow{CA}$, $\overrightarrow{CB}$, $\overrightarrow{CC}$.

2- Calculate the moment of the system of the three forces about C.

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3- If the moment of the system is zero, deduce the value of the constant a.

4- Calculate the moment of the system of the three forces about O, and find its magnitude.

Exercise 5

In an orthonormal basis $(\vec{\imath},\ \vec{\jmath},\ \vec{k})$, we give the vectors: $\begin{cases} \vec{u} = 2t\,\vec{\imath} + (t+1)\vec{\jmath} \\ \vec{w} = 4t\,\vec{\imath} - 3t\,\vec{\jmath} + 2\,\vec{k} \end{cases}$

Calculate the derivatives: $\dfrac{d\vec{u}}{dt}$, $\dfrac{d\vec{w}}{dt}$, $\dfrac{d\vec{u}\bullet\vec{w}}{dt}$ and $\dfrac{d\vec{u}\wedge\vec{w}}{dt}$

Exercise 6

In an orthonormal basis $(\vec{\imath},\ \vec{\jmath})$, we give the following vectors: $\begin{cases} \vec{r} = R\cos\theta\ \vec{\imath} + R\sin\theta\ \vec{\jmath} \quad ; R: C^t \\ \overrightarrow{u_r} = \cos\theta\ \vec{\imath} + \sin\theta\ \vec{\jmath} \end{cases}$

1- Find the derivative vector: $\overrightarrow{u_\theta} = \dfrac{d\overrightarrow{u_r}}{d\theta}$.

2- Show that $\overrightarrow{u_r}$ and $\overrightarrow{u_\theta}$ are unit vectors.

3- Show that the vectors $\overrightarrow{u_r}$ and $\overrightarrow{u_\theta}$ are perpendicular.

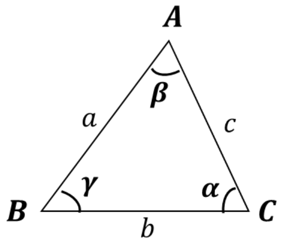
4- Show that the two vectors $\vec{r}$ and $\overrightarrow{u_r}$ are collinear (parallel).

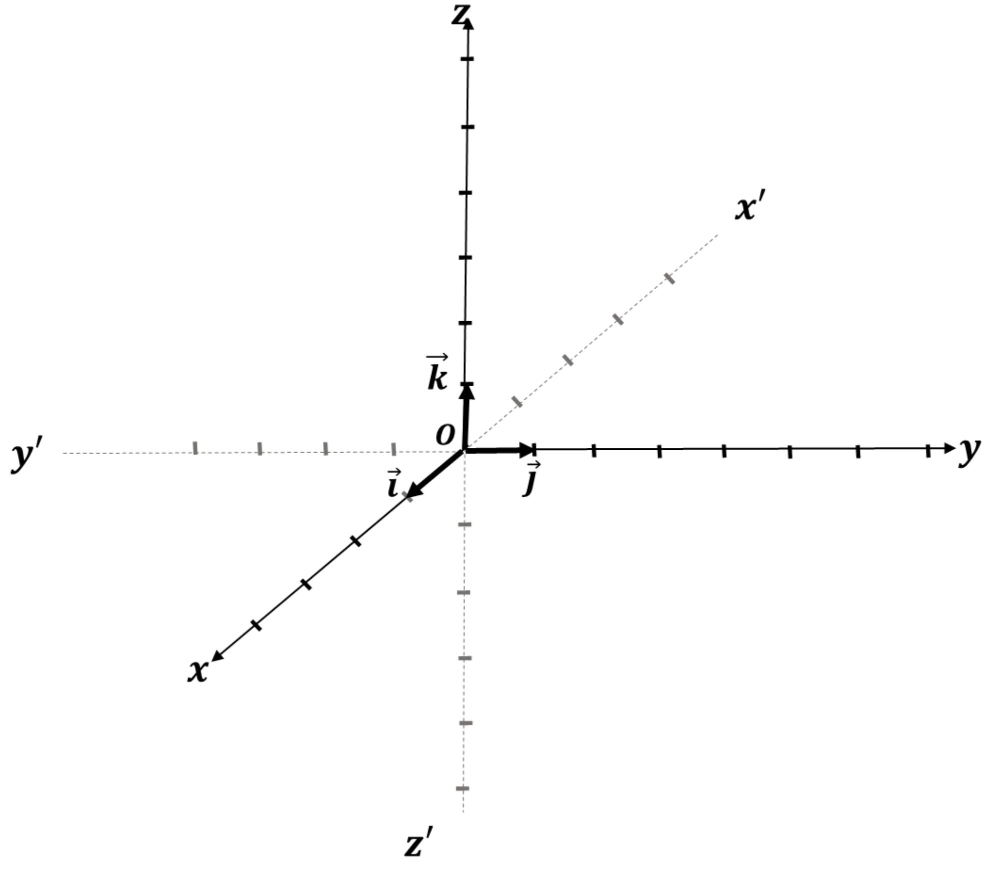
5- What can you say about vectors $(\overrightarrow{u_r}, \overrightarrow{u_\theta})$.

Exercise 7 (Homework)

We consider a triangle ABC with sides a, b and c and angles α, β, γ.

Prove the following sines law: $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$





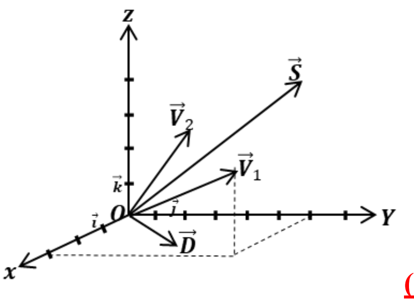
Exercise 01

$$\vec{V_1} = 3\,\vec{\imath} + 6\,\vec{\jmath} + 3\,\vec{k}$$
$$\vec{V_2} = -\vec{\imath} + \vec{\jmath} + 2\,\vec{k}$$

1) **Sum and Difference vectors $\vec{S}$ and $\vec{D}$**

$$\vec{S} = \vec{V_1} + \vec{V_2} = 2\,\vec{\imath} + 7\,\vec{\jmath} + 5\,\vec{k}$$
$$\vec{D} = \vec{V_1} - \vec{V_2} = 4\,\vec{\imath} + 5\,\vec{\jmath} + \vec{k}$$

2) **Presentation of vectors in 3D-space (Oxyz)**



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3) **Scalar and vector product**

$$\vec{V_1} \bullet \vec{V_2} = 9$$

$$\vec{V_1} \wedge \vec{V_2} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ 3 & 6 & 3 \\ -1 & 1 & 2 \end{vmatrix}$$

$$\vec{V_1} \wedge \vec{V_2} = 9\,\vec{\imath} - 9\,\vec{\jmath} + 9\,\vec{k}$$

4) **Angle α between $\vec{V_1}$ and $\vec{V_2}$**

$$\|\vec{V_1}\| = 3\sqrt{6}$$
$$\|\vec{V_2}\| = \sqrt{6}$$

$$\cos\alpha = \frac{\vec{V_1} \bullet \vec{V_2}}{\|\vec{V_1}\|\,\|\vec{V_2}\|} = \frac{1}{2}\ ; \qquad \alpha = 60\,°$$

5) **Projection of vectors $\vec{V_1}$ on $\vec{V_2}$ and $\vec{V_2}$ on $\vec{V_1}$**

$$\begin{cases} Proj\ \vec{V_1}/\vec{V_2} = \dfrac{\vec{V_1} \bullet \vec{V_2}}{\|\vec{V_2}\|} = \dfrac{3\sqrt{6}}{2} \\ Proj\ \vec{V_2}/\vec{V_1} = \dfrac{\vec{V_1} \bullet \vec{V_2}}{\|\vec{V_1}\|} = \dfrac{\sqrt{6}}{2} \end{cases}$$

Exercise -02

1) Vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$

Generally:

$$\overrightarrow{AB} = (x_B - x_A)\,\vec{\imath} + (y_B - y_A)\,\vec{\jmath} + (z_B - z_A)\vec{k}$$

$$\begin{cases} \overrightarrow{AB} = B - A = -3\,\vec{\imath} + \vec{\jmath} + 2\,\vec{k} \\ \overrightarrow{AC} = C - A = -\vec{\imath} + 3\,\vec{\jmath} + 3\,\vec{k} \\ \overrightarrow{BC} = C - B = 2\,\vec{\imath} + 2\,\vec{\jmath} + \vec{k} \end{cases}$$

2) Magnitudes $\|\overrightarrow{AB}\|$, $\|\overrightarrow{AC}\|$, and $\|\overrightarrow{BC}\|$

$\|\overrightarrow{AB}\| = \sqrt{14}$, $\|\overrightarrow{AC}\| = \sqrt{19}$, and $\|\overrightarrow{BC}\| = 3$

3) Unit vectors $\overrightarrow{u_{AB}}$, $\overrightarrow{u_{AC}}$, and $\overrightarrow{u_{BC}}$

We have:

$$\vec{u} = \frac{\vec{V}}{\|\vec{V}\|}$$

$$\begin{cases} \overrightarrow{U_{AB}} = \dfrac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \dfrac{\sqrt{14}}{14}\left(-3\,\vec{\imath} + \vec{\jmath} + 2\,\vec{k}\right) \\ \overrightarrow{U_{AC}} = \dfrac{\overrightarrow{AC}}{\|\overrightarrow{AC}\|} = \dfrac{\sqrt{19}}{19}\left(-\vec{\imath} + 3\,\vec{\jmath} + 3\,\vec{k}\right) \\ \overrightarrow{U_{BC}} = \dfrac{\overrightarrow{BC}}{\|\overrightarrow{BC}\|} = \dfrac{1}{3}\left(2\,\vec{\imath} + 2\,\vec{\jmath} + \vec{k}\right) \end{cases}$$

4) Surface area of triangle ABC

$$\overrightarrow{AB} \wedge \overrightarrow{AC} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ -3 & 1 & 2 \\ -1 & 3 & 3 \end{vmatrix} = -3\,\vec{\imath} + 7\,\vec{\jmath} - 8\,\vec{k}$$

Surface area of triangle ABC:

$$S_T = \frac{\|\overrightarrow{AB} \wedge \overrightarrow{AC}\|}{2} = \frac{\sqrt{122}}{2}\ (u.s)$$

Exercise -03

$$A\,(2, -3, 1);\ B\,(5, 5, -4)\ ;\ C\,(0, 0, 4)$$

1) **Vector $\vec{S} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$**

$$\vec{S} = 7\,\vec{\imath} + 2\,\vec{\jmath} + \vec{k}$$

2) **The directional cosines of the vector $\vec{S}$**

We have: $\vec{u} = \cos\alpha\ \vec{\imath} + \cos\beta\ \vec{\jmath} + \cos\gamma\ \vec{k}$

and: $\|\vec{S}\| = \sqrt{54} = 3\sqrt{6}$

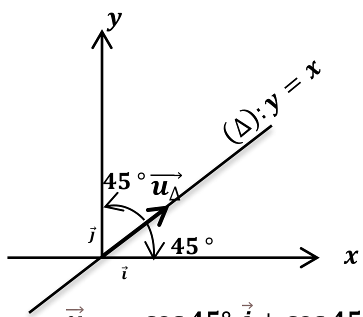
$$\vec{u} = \frac{\vec{S}}{\|\vec{S}\|} = \frac{7}{3\sqrt{6}}\,\vec{\imath} + \frac{2}{3\sqrt{6}}\,\vec{\jmath} + \frac{1}{3\sqrt{6}}\,\vec{k}$$

$$\begin{cases} \cos\alpha = \dfrac{7}{3\sqrt{6}} \\ \cos\beta = \dfrac{2}{3\sqrt{6}} \\ \cos\gamma = \dfrac{1}{3\sqrt{6}} \end{cases}$$

3) **Projection of the vector $\vec{S}$ on the axes**

$$\begin{cases} Proj\ \vec{S}/_{Ox} = \dfrac{\vec{S}\,.\,\vec{\imath}}{\|\vec{\imath}\|} = 7 \\ Proj\ \vec{S}/_{Oy} = \dfrac{\vec{S}\,.\,\vec{\jmath}}{\|\vec{\jmath}\|} = 2 \\ Proj\ \vec{S}/_{Oz} = \dfrac{\vec{S}\,.\,\vec{k}}{\|\vec{k}\|} = 1 \end{cases}$$

For straight line (Δ) : $y = x$



$\vec{u}_\Delta = \cos 45°\ \vec{i} + \cos 45°\ \vec{j}$

$$\Rightarrow \vec{u}_\Delta = \frac{\sqrt{2}}{2}\ \vec{\imath} + \frac{\sqrt{2}}{2}\ \vec{\jmath}\ ;\ \|\vec{u}_\Delta\| = 1$$

$$Proj\ {}^{\vec{S}}/_{\Delta} = \frac{\vec{S}\ .\ \vec{u}_\Delta}{\|\vec{u}_\Delta\|} = \frac{9\sqrt{2}}{2}$$

4) **Vector product $\overrightarrow{OB} \wedge \overrightarrow{OC}$; The surface of a triangle**

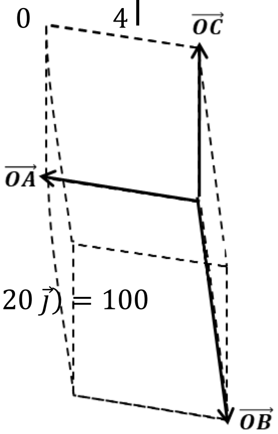
$$\overrightarrow{OB} \wedge \overrightarrow{OC} = 20\,\vec{\imath} - 20\,\vec{\jmath}$$

$$S_T = \frac{\|\overrightarrow{OB} \wedge \overrightarrow{OC}\|}{2} = 10\sqrt{2}\ (u.s)$$

5) **Mixed product $\overrightarrow{OA} \bullet (\overrightarrow{OB} \wedge \overrightarrow{OC})$**

Method 1:

$$\overrightarrow{OA} \bullet (\overrightarrow{OB} \wedge \overrightarrow{OC}) = \begin{vmatrix} 2 & -(-3) & 1 \\ 5 & 5 & -4 \\ 0 & 0 & 4 \end{vmatrix} = 100$$



Method 2 :

$$\overrightarrow{OA} \bullet (\overrightarrow{OB} \wedge \overrightarrow{OC}) =$$

$$(2\,\vec{\imath} - 3\,\vec{\jmath} + \vec{k}) \bullet (20\,\vec{\imath} - 20\,\vec{\jmath}) = 100$$

$$V_p = 100\ (u.v)$$

6) **The moment of the vector $\vec{S}$ about a point $I\ (1, 0, 4)$**

$$\vec{M}_I(\vec{S}) = \overrightarrow{IO} \wedge \vec{S} = -\overrightarrow{OI} \wedge \vec{S} \qquad ; \quad (\vec{S} \equiv \overrightarrow{OS})$$

$$\vec{M}_I(\vec{S}) = 8\,\vec{\imath} - 27\vec{\jmath} - 2\,\vec{k}$$

7) **The moment of the vector $\vec{S}$ about a straight line (IC)**

We have: $\boldsymbol{M_{IC}(\vec{S}) = \vec{M}_I(\vec{A}) \bullet \overrightarrow{u_{IC}}}$

Thus: $\overrightarrow{IC} = (0-1)\vec{\imath} + (0-0)\vec{\jmath} + (4-4)\vec{k}$

$$\overrightarrow{IC} = -\vec{\imath}$$

$$\|\overrightarrow{IC}\| = 1 \qquad \Rightarrow \overrightarrow{u_{IC}} = \frac{\overrightarrow{IC}}{\|\overrightarrow{IC}\|} = -\vec{\imath}$$

$$M_{IC}(\vec{S}) = (8\,\vec{\imath} - 27\vec{\jmath} - 2\,\vec{k}) \bullet (-\vec{\imath}) = -8$$

Exercise -04

1) The vectors: $\overrightarrow{CA}, \overrightarrow{CB}, \overrightarrow{CC}$

$$\begin{cases} \overrightarrow{CA} = A - C = -2\,\vec{\jmath} + (1+a)\vec{k} \\ \overrightarrow{CB} = B - C = 5\,\vec{\imath} + 2 + 9\,\vec{k} \\ \overrightarrow{CC} = C - C = \vec{0} \end{cases}$$

2) Moment of the system of the three forces about C

$$\vec{M}_C(\vec{F}) = \sum \vec{M}_C(\vec{F}_i) = \vec{M}_C(\vec{F}_1) + \vec{M}_C(\vec{F}_2) + \vec{M}_C(\vec{F}_3)$$

$$\vec{M}_C(\vec{F}) = \overrightarrow{CA} \wedge \vec{F}_1 + \overrightarrow{CB} \wedge \vec{F}_2 + \overrightarrow{CC} \wedge \vec{F}_3$$

$$\vec{M}_C(\vec{F}) = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ 0 & -2 & (1+a) \\ 3 & 1 & 4 \end{vmatrix} + \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ 5 & 2 & 9 \\ \frac{1}{2} & -1 & \frac{a}{2} \end{vmatrix} + 0$$

$$\vec{M}_C(\vec{F}) = \frac{1}{2}(-11a + 3)\vec{\jmath}$$

3) Value of the constant a

The moment of the system is zero:

$$\vec{M}_C(\vec{F}) = \frac{1}{2}(a + 15)\vec{\jmath} = \vec{0}$$

$$a = \frac{3}{11}$$

4) Moment of the system of the three forces about O

$$\vec{M}_O(\vec{F}) = \sum \vec{M}_O(\vec{F}_i) = \vec{M}_O(\vec{F}_1) + \vec{M}_O(\vec{F}_2) + \vec{M}_O(\vec{F}_3)$$

$$\vec{M}_O(\vec{F}) = \overrightarrow{OA} \wedge \vec{F}_1 + \overrightarrow{OB} \wedge \vec{F}_2 + \overrightarrow{OC} \wedge \vec{F}_3$$

$$\vec{M}_O(\vec{F}) = 2\vec{\imath} + 2\vec{k}$$

Magnitude: $\vec{M}_O(\vec{F}) = 2\sqrt{2}$

Exercise -05

Calculate the derivatives:

- $\frac{d\vec{u}}{dt} = 2\vec{\imath} + \vec{\jmath}$
- $\frac{d\vec{w}}{dt} = 4\vec{\imath} - 3\vec{\jmath}$
- $\frac{d\vec{u} \bullet \vec{w}}{dt} = \frac{d\vec{u}}{dt} \bullet \vec{w} + \vec{u} \bullet \frac{d\vec{w}}{dt} = 10t - 3$
- $\frac{d\vec{u} \wedge \vec{w}}{dt} = \frac{d\vec{u}}{dt} \wedge \vec{w} + \vec{u} \wedge \frac{d\vec{w}}{dt} = 2\,\vec{\imath} - 4\vec{\jmath} - 4(3t+1)\,\vec{k}$

Exercise -06

1) The vector $\overrightarrow{u_\theta}$

$$\overrightarrow{u_\theta} = \frac{d\overrightarrow{u_r}}{d\theta} = -\sin\theta\ \vec{\imath} + \cos\theta\ \vec{\jmath}$$

2) The unit vectors

For $\overrightarrow{u_r}$ and $\overrightarrow{u_\theta}$ to be unit vectors, their magnitude (norm) must be

$$\begin{cases} \|\overrightarrow{u_r}\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1 \\ \|\overrightarrow{u_\theta}\| = \sqrt{\sin^2\theta + \cos^2\theta} = 1 \end{cases}$$

So: $\overrightarrow{u_r}$ and $\overrightarrow{u_\theta}$ are unit vectors.

1) **<u>Show that $\overrightarrow{u_r} \perp \overrightarrow{u_\theta}$</u>**

For $\overrightarrow{u_r}$ and $\overrightarrow{u_\theta}$ to be orthogonal (perpendicular), their scalar product must be **<u>null</u>** (zero).

$$\overrightarrow{u_r} \bullet \overrightarrow{u_\theta} = -\cos\theta \sin\theta + \sin\theta \cos\theta = 0$$

Thus: $\overrightarrow{u_r} \perp \overrightarrow{u_\theta}$

2) **<u>Show that $\vec{r} \parallel \overrightarrow{u_r}$</u>**

For $\vec{r}$ and $\overrightarrow{u_r}$ to be collinear (parallel), it is necessary that $\vec{r} = k\,\overrightarrow{u_r}$: k is C^t.

We have : $\vec{r} = R\,(\cos\theta\ \vec{\imath} + \sin\theta\ \vec{\jmath})$

$\Rightarrow \boldsymbol{\vec{r} = R\,\overrightarrow{u_r}}$

Thus : $\vec{r} \parallel \overrightarrow{u_r}$

3) **We say that the vectors $(\overrightarrow{u_r}, \overrightarrow{u_\theta})$ can form an orthonormal basis, which is the base of the polar coordinate system.**

<u>Exercise -07 (HW)</u>

Let us show the following law: $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$

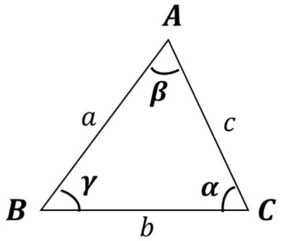
$$S_T = \frac{\|\overrightarrow{CA} \wedge \overrightarrow{CB}\|}{2} = \frac{b.c.\sin\alpha}{2} \qquad (1)$$

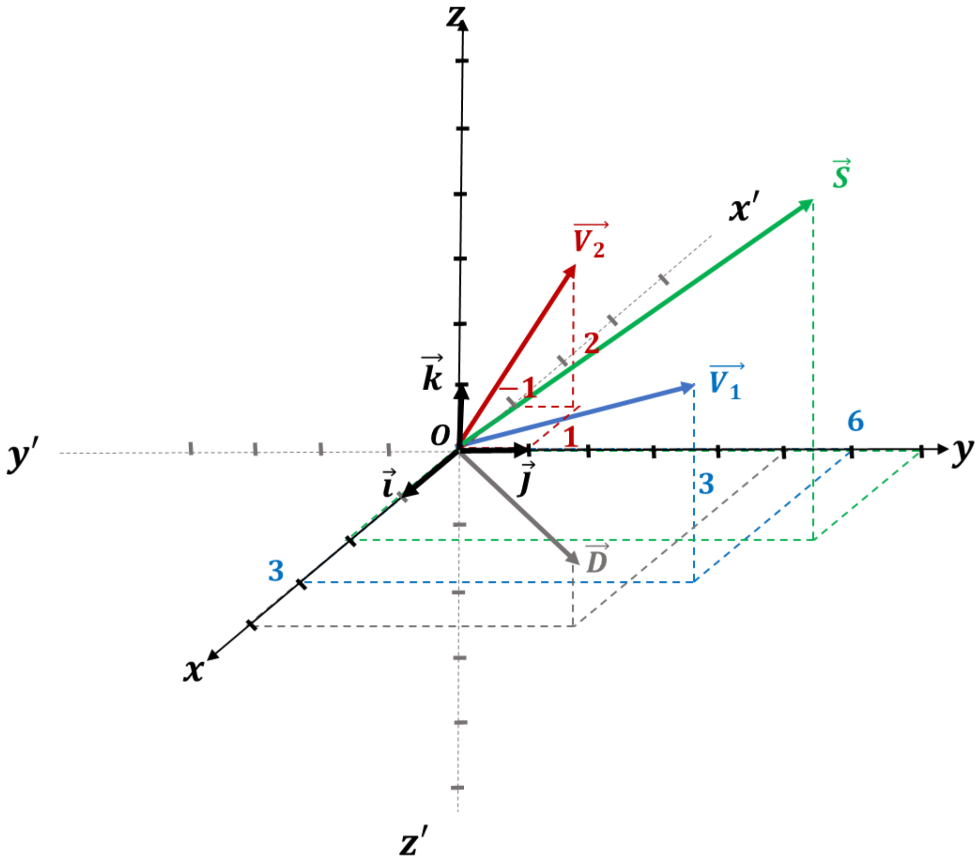
$$S_T = \frac{\|\overrightarrow{AB} \wedge \overrightarrow{AC}\|}{2} = \frac{a.c.\sin\beta}{2} \qquad (2)$$

$$S_T = \frac{\|\overrightarrow{BC} \wedge \overrightarrow{BA}\|}{2} = \frac{a.b.\sin\gamma}{2} \qquad (3)$$

According to the equality between (1,2,3) and their division on the product a.b.c, we obtain:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$





z
$\vec{S}$
x'
$\vec{V_2}$
2
$\vec{k}$
−1
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O
1
6
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y
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3
3
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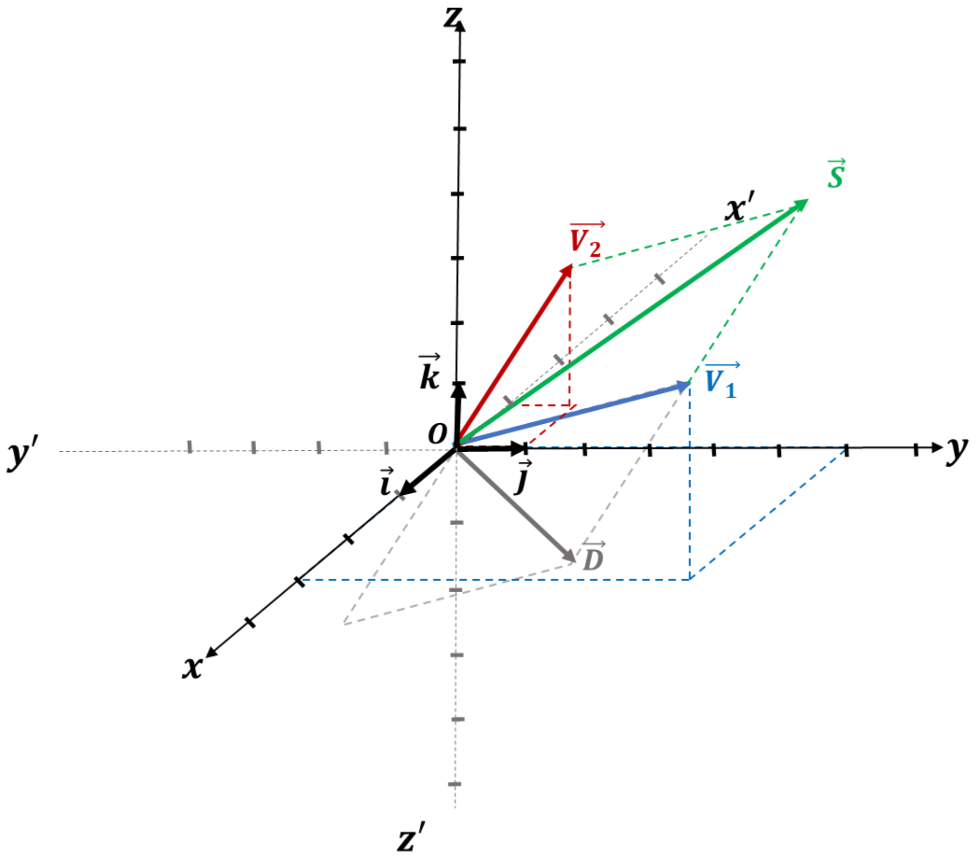
x
z'



z
$\vec{S}$
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z'

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