11.7 IMPROPER INTEGRALS

1. Integrals with infinite limits. Let a function $f(x)$ be defined and continuous for all values of x such that $a \leqslant x < +\infty$. Consider the integral

$$I(b) = \int_a^b f(x)\,dx$$

This integral is meaningful for any $b > a$. The integral varies with b and is a continuous function of b (see Sec. 11.4). Let us consider the behaviour of this integral when $b \to +\infty$ (Fig. 222).

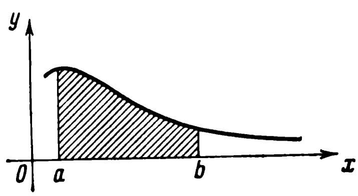

Fig. 222

Definition. If there exists a finite limit

$$\lim_{b \to +\infty} \int_a^b f(x)\,dx$$

then this limit is called the *improper integral* of the function $f(x)$ on the interval $[a, +\infty)$ and is denoted by the symbol

$$\int_a^{+\infty} f(x)\,dx$$

Thus, by definition, we have

$$\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx$$

In this case it is said that the improper integral $\int\limits_a^{+\infty} f(x)\,dx$ *exists* or *converges*. If $\int\limits_a^b f(x)\,dx$ as $b \to +\infty$ does not have a finite limit, one says that $\int\limits_a^{+\infty} f(x)\,dx$ *does not exist* or *diverges*.

It is easy to see the geometric meaning of an improper integral for the case where $f(x) \geqslant 0$: if the integral $\int\limits_a^b f(x)\,dx$ expresses the area of a region bounded by the curve $y=f(x)$, the x-axis and the ordinates $x=a$, $x=b$, it is natural to consider that the improper integral $\int\limits_a^{+\infty} f(x)\,dx$ expresses the area of an unbounded (infinite) region lying between the lines $y=f(x)$, $x=a$, and the axis of abscissas.

We similarly define the improper integrals of other infinite intervals:

$$\int\limits_{-\infty}^{a} f(x)\,dx = \lim_{\alpha \to -\infty} \int\limits_{\alpha}^{a} f(x)\,dx$$

$$\int\limits_{-\infty}^{+\infty} f(x)\,dx = \int\limits_{-\infty}^{c} f(x)\,dx + \int\limits_{c}^{+\infty} f(x)\,dx$$

The latter equation should be understood as follows: if each of the improper integrals on the right exists, then, by definition, the integral on the left also exists (converges).

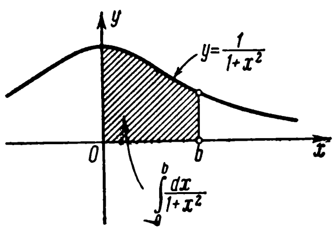

Fig. 223

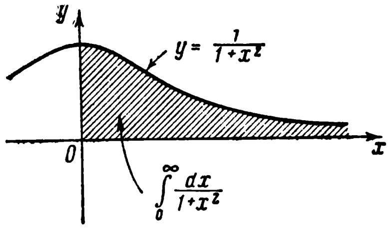

Fig. 224

Example 1. Evaluate the integral $\int\limits_0^{+\infty} \frac{dx}{1+x^2}$ (see Figs. 223 and 224).

Solution. By the definition of an improper integral we find

$$\int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{b \to +\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b \to +\infty} \arctan x \Big|_0^b = \lim_{b \to +\infty} \arctan b = \frac{\pi}{2}$$

This integral expresses the area of an infinite curvilinear trapezoid cross-hatched in Fig. 224.

Example 2. Find out at which values of α (Fig. 225) the integral

$$\int_1^{+\infty} \frac{dx}{x^\alpha}$$

converges and at which it diverges.

Solution. Since (when $\alpha \neq 1$)

$$\int_1^b \frac{dx}{x^\alpha} = \frac{1}{1-\alpha} x^{1-\alpha} \Big|_1^b = \frac{1}{1-\alpha} (b^{1-\alpha} - 1)$$

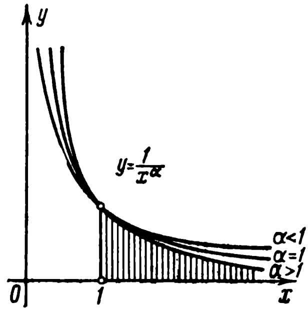

Fig. 225

we have

$$\int_1^{+\infty} \frac{dx}{x^\alpha} = \lim_{b \to +\infty} \frac{1}{1-\alpha} (b^{1-\alpha} - 1)$$

Consequently, with respect to this integral we conclude that

if $\alpha > 1$, then $\int_1^{+\infty} \frac{dx}{x^\alpha} = \frac{1}{\alpha - 1}$, and the integral converges;

if $\alpha < 1$, then $\int_1^{+\infty} \frac{dx}{x^\alpha} = \infty$, and the integral diverges

When $\alpha = 1$, $\int_1^{+\infty} \frac{dx}{x} = \ln x \Big|_1^{+\infty} = \infty$, and the integral diverges.

Example 3. Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

Solution.

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{+\infty} \frac{dx}{1+x^2}$$

The second integral is equal to $\frac{\pi}{2}$ (see Example 1). Compute the first integral:

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{\alpha \to -\infty} \int_{\alpha}^{0} \frac{dx}{1+x^2} = \lim_{\alpha \to -\infty} \arctan x \Big|_{\alpha}^{0}$$

$$= \lim_{\alpha \to -\infty} (\arctan 0 - \arctan \alpha) = \frac{\pi}{2}$$

Therefore,

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

In many cases it is sufficient to determine whether the given integral converges or diverges, and to estimate its value. The following theorems, which we give without proof, may be useful in this respect. We shall illustrate their application in a few cases.

Theorem 1. *If for all $x\ (x \geqslant a)$ the inequality*

$$0 \leqslant f(x) \leqslant \varphi(x)$$

is fulfilled and if $\int_{a}^{+\infty} \varphi(x)\,dx$ *converges, then* $\int_{a}^{+\infty} f(x)\,dx$ *also converges, and*

$$\int_{a}^{+\infty} f(x)\,dx \leqslant \int_{a}^{+\infty} \varphi(x)\,dx$$

Example 4. Investigate the integral

$$\int_{1}^{+\infty} \frac{dx}{x^2(1+e^x)}$$

for convergence.

Solution. It will be noted that when $1 \leqslant x$,

$$\frac{1}{x^2(1+e^x)} < \frac{1}{x^2}$$

And

$$\int_{1}^{+\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{1}^{+\infty} = 1$$

Consequently,

$$\int_{1}^{+\infty} \frac{dx}{x^2(1+e^x)}$$

converges, and its value is less than 1.

Theorem 2. *If for all $x\,(x \geqslant a)$ the inequality $0 \leqslant \varphi(x) \leqslant f(x)$ holds true and $\int_a^{+\infty} \varphi(x)\,dx$ diverges, then the integral $\int_a^{+\infty} f(x)\,dx$ also diverges.*

Example 5. Find out whether the following integral converges or diverges:

$$\int_1^{+\infty} \frac{x+1}{\sqrt{x^3}}\,dx$$

We notice that

$$\frac{x+1}{\sqrt{x^3}} > \frac{x}{\sqrt{x^3}} = \frac{1}{\sqrt{x}}$$

But

$$\int_1^{+\infty} \frac{dx}{\sqrt{x}} = \lim_{b \to +\infty} 2\sqrt{x}\,\Big|_1^b = +\infty$$

Consequently, the given integral also diverges.

In the last two theorems we considered improper integrals of nonnegative functions. For the case of a function $f(x)$ which changes its sign over an infinite interval we have the following theorem.

Theorem 3. *If the integral $\int_a^{+\infty} |f(x)|\,dx$ converges, then the integral $\int_a^{+\infty} f(x)\,dx$ also converges.*

In this case, the latter integral is called an *absolutely convergent integral.*

Example 6. Investigate the convergence of the integral

$$\int_1^{+\infty} \frac{\sin x}{x^3}\,dx$$

Solution. Here, the integrand is an alternating function. We note that

$$\left|\frac{\sin x}{x^3}\right| \leqslant \left|\frac{1}{x^3}\right|. \quad \text{But} \quad \int_1^{+\infty} \frac{dx}{x^3} = -\frac{1}{2x^2}\Big|_1^{+\infty} = \frac{1}{2}$$

Therefore, the integral $\int_1^{+\infty} \left|\frac{\sin x}{x^3}\right| dx$ converges. Whence it follows that the given integral also converges.

three intermediate points) is in better agreement with the true value of the integral than the result obtained by the trapezoidal rule (with nine intermediate points).

The theory of approximation of integrals was further developed in the works of Academician A. N. Krylov (1863-1945).

11.10 INTEGRALS DEPENDENT ON A PARAMETER. THE GAMMA FUNCTION

Differentiating integrals dependent on a parameter. Suppose we have an integral

$$I(\alpha)=\int_a^b f(x,\ \alpha)\,dx \tag{1}$$

in which the integrand is dependent upon some parameter α. If the parameter α varies, then the value of the definite integral will also vary. Thus the definite integral is a **function** of α; we can therefore denote it by $I(\alpha)$.

1. Suppose that $f(x,\ \alpha)$ and $f'_\alpha(x,\ \alpha)$ are continuous functions when

$$c\leqslant\alpha\leqslant d \text{ and } a\leqslant x\leqslant b \tag{2}$$

Find the derivative of the integral with respect to the parameter α:

$$\lim_{\Delta\alpha\to 0}\frac{I(\alpha+\Delta\alpha)-I(\alpha)}{\Delta\alpha}=I'_\alpha(\alpha)$$

In finding this derivative we note that

$$\frac{I(\alpha+\Delta\alpha)-I(\alpha)}{\Delta\alpha}=\frac{1}{\Delta\alpha}\left[\int_a^b f(x,\ \alpha+\Delta\alpha)\,dx-\int_a^b f(x,\ \alpha)\,dx\right]$$

$$=\int_a^b\frac{f(x,\ \alpha+\Delta\alpha)-f(x,\ \alpha)}{\Delta\alpha}\,dx$$

Applying the Lagrange theorem to the integrand we have

$$\frac{f(x,\ \alpha+\Delta\alpha)-f(x,\ \alpha)}{\Delta\alpha}=f'_\alpha(x,\ \alpha+\theta\,\Delta\alpha)$$

where $0<\theta<1$. Since $f'_\alpha(x,\ \alpha)$ is continuous in the closed domain (2), we have

$$f'_\alpha(x,\ \alpha+\theta\,\Delta\alpha)=f'_\alpha(x,\ \alpha)+\varepsilon$$

where the quantity ε, which depends on x, α, $\Delta\alpha$, approaches zero as $\Delta\alpha\to 0$.

Thus,

$$\frac{I(\alpha+\Delta\alpha)-I(\alpha)}{\Delta\alpha}=\int_a^b [f'_\alpha(x,\ \alpha)+\varepsilon]\,dx=\int_a^b f'_\alpha(x,\ \alpha)\,dx+\int_a^b \varepsilon\,dx$$

Passing to the limit as $\Delta\alpha\to 0$, we have*

$$\lim_{\Delta\alpha\to 0}\frac{I(\alpha+\Delta\alpha)-I(\alpha)}{\Delta\alpha}=I'_\alpha(\alpha)=\int_a^b f'_\alpha(x,\ \alpha)\,dx$$

or

$$\left[\int_a^b f(x,\ \alpha)\,dx\right]'_\alpha=\int_a^b f'_\alpha(x,\ \alpha)\,dx$$

This formula is called the *Leibniz formula*.

2. Now suppose that in the integral (1) the *limits of integration a and b are functions of* α:

$$I(\alpha)=\Phi[\alpha,\ a(\alpha),\ b(\alpha)]=\int_{a(\alpha)}^{b(\alpha)} f(x,\ \alpha)\,dx \qquad (1')$$

$\Phi[\alpha,\ a(\alpha),\ b(\alpha)]$ is a composite function of α, and a and b are intermediate arguments. To find the derivative of $I(\alpha)$, apply the rule for differentiating a composite function of several variables (see Sec. 8.10):

$$I'(\alpha)=\frac{\partial\Phi}{\partial\alpha}+\frac{\partial\Phi}{\partial a}\frac{da}{d\alpha}+\frac{d\Phi}{\partial b}\frac{db}{d\alpha} \qquad (3)$$

By the theorem on the differentiation of a definite integral with respect to a variable upper limit (see Sec. 11.4) we get

$$\frac{\partial\Phi}{\partial b}=\frac{\partial}{\partial b}\int_a^b f(x,\ \alpha)\,dx=f[b(\alpha),\ \alpha]$$

$$\frac{\partial\Phi}{\partial a}=\frac{\partial}{\partial a}\int_a^b f(x,\ \alpha)\,dx=-\frac{\partial}{\partial a}\int_b^a f(x,\ \alpha)\,dx=-f[a(\alpha),\ \alpha]$$

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* The integrand in the integral $I=\int_a^b \varepsilon\,d\alpha$ approaches zero as $\Delta\alpha\to 0$. From the fact that the integrand approaches zero at each point it does not always follow that the integral also approaches zero. However, in the given case, I approaches zero as $\Delta\alpha\to 0$. We accept this fact without proof.

Finally, to evaluate $\frac{\partial \Phi}{\partial \alpha}$ use the above-derived Leibniz formula:

$$\frac{\partial \Phi}{\partial \alpha}=\int_{a}^{b} f'_{\alpha}(x, \alpha)\, dx.$$

Substituting into (3) the expressions obtained for the derivatives, we have

$$I'_{\alpha}(\alpha)=\int_{a(\alpha)}^{b(\alpha)} f'_{\alpha}(x, \alpha)\, dx+f[b(\alpha), \alpha]\frac{db}{d\alpha}-f[a(\alpha), \alpha]\frac{da}{d\alpha} \quad (4)$$

Using the Leibniz formula it is possible to compute certain definite integrals.

Example. Evaluate the integral

$$I(\alpha)=\int_{0}^{\infty} e^{-x}\frac{\sin \alpha x}{x}\, dx$$

Solution. First note that it is impossible to compute the integral directly, because the antiderivative of the function $e^{-x}\frac{\sin \alpha x}{x}$ is not expressible in terms of elementary functions. To compute this integral we shall consider it as a function of the parameter α. Then its derivative with respect to α is found from the above-derived Leibniz formula *:

$$I'(\alpha)=\int_{0}^{\infty}\left[e^{-x}\frac{\sin \alpha x}{x}\right]'_{\alpha} dx=\int_{0}^{\infty} e^{-x}\cos \alpha x\, dx$$

But the latter integral is readily evaluated by means of elementary functions; it is equal to $\frac{1}{1+\alpha^2}$. Therefore,

$$I'(\alpha)=\frac{1}{1+\alpha^2}$$

Integrating the identity obtained, we find $I(\alpha)$:

$$I(\alpha)=\arctan \alpha+C$$

We have C to determine now. To do this, we note that

$$I(0)=\int_{0}^{\infty} e^{-x}\frac{\sin 0\cdot x}{x}\, dx=\int_{0}^{\infty} 0\, dx=0$$

* Leibniz' formula was derived on the assumption that the limits of integration a and b are finite. However, in this case Leibniz' formula also holds, even though one of the limits of integration is equal to infinity. For the conditions under which differentiation of improper integrals with respect to a parameter is permissible. See G. M. Fikhtengolts, *Course of Differential and Integral Calculus*, Fizmatgiz, 1962, Vol. II, Ch. XIV, Sec. 3 (in Russian).

Besides, arctan $0=0$. Substituting into (5) $\alpha=0$, we get

$$I(0)=\arctan 0+C$$

whence $C=0$. Hence, for any value of α we have $I(\alpha)=\arctan\alpha$, that is,

$$\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}\,dx=\arctan\alpha$$

Example 2. The gamma function.

We consider an integral dependent on a parameter α,

$$\int_0^\infty x^{\alpha-1}e^{-x}\,dx \tag{6}$$

and we will show that this improper integral exists (converges) for $\alpha>0$. We represent it in the form of a sum

$$\int_0^\infty x^{\alpha-1}e^{-x}\,dx=\int_0^1 x^{\alpha-1}e^{-x}\,dx+\int_1^\infty x^{\alpha-1}e^{-x}\,dx$$

The first integral on the right converges, since

$$0<\int_0^1 x^{\alpha-1}e^{-x}\,dx<\int_0^1 x^{\alpha-1}\,dx=\frac{1}{\alpha}$$

The second integral likewise converges. Indeed, let n be an integer such that $n>\alpha-1$. Then clearly

$$0<\int_0^\infty x^{\alpha-1}e^{-x}\,dx<\int_1^\infty x^n e^{-x}\,dx<\infty$$

Integrate the latter integral by parts noting that

$$\lim_{x\to+\infty}\frac{x^k}{e^x}=0 \tag{7}$$

for an arbitrary positive integer k. Thus, integral (6) defines a certain function α. This function is denoted by $\Gamma(\alpha)$ and is called the *gamma function*:

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx \tag{8}$$

It is widely used in applied mathematics. Let us find the values of $\Gamma(\alpha)$ for integral α. For $\alpha=1$ we have

$$\Gamma(1)=\int_0^\infty e^{-x}\,dx=1 \tag{9}$$

Let the integer $\alpha>1$. We integrate by parts:

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx=-x^{\alpha-1}e^{-x}\Big|_0^\infty+(\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}\,dx$$

or, taking into account (7),

$$\Gamma(\alpha)=(\alpha-1)\,\Gamma(\alpha-1) \tag{10}$$

By (10) and (9), we find that for $\alpha=n$

$$\Gamma(n)=(n-1)! \tag{11}$$

11.11 INTEGRATION OF A COMPLEX FUNCTION OF A REAL VARIABLE

In Sec. 7.4 we defined a complex function $\tilde{f}(x)=u(x)+iv(x)$ of a real variable x and also its derivative $\tilde{f}'(x)=u'(x)+iv'(x)$.

Definition. A function $\tilde{F}(x)=U(x)+iV(x)$ is called an *antiderivative of a complex function of a real variable* $\tilde{f}(x)$ if

$$\tilde{F}'(x)=\tilde{f}(x) \tag{1}$$

that is, if

$$U'(x)+iV'(x)=u(x)+iv(x) \tag{2}$$

From (2) it follows that $U'(x)=u(x)$, $V'(x)=v(x)$, that is, $U(x)$ is an antiderivative of $u(x)$ and $V(x)$ is an antiderivative of $v(x)$.

It follows, from this definition and from the remark, that if $\tilde{F}(x)=U(x)+iV(x)$ is an antiderivative of the function $\tilde{f}(x)$, then any antiderivative of $\tilde{f}(x)$ is of the form $\tilde{F}(x)+C$, where C is an arbitrary complex constant. We will call the expression $\tilde{F}(x)+C$ the *indefinite integral of a complex function of a real variable* and we will write

$$\int \tilde{f}(x)\,dx=\int u(x)\,dx+i\int v(x)\,dx=\tilde{F}(x)+C \tag{3}$$

The *definite integral* of a complex function of a real variable, $\tilde{f}(x)=u(x)+iv(x)$, is defined as follows:

$$\int_a^b \tilde{f}(x)\,dx=\int_a^b u(x)\,dx+i\int_a^b v(x)\,dx \tag{4}$$

This definition does not contradict and is in full agreement with the definition of the definite integral as the limit of a sum.

Exercises on Chapter 11

1. Form the integral sum s_n and pass to the limit to compute the following definite integrals $\int_a^b x^2\,dx$. **Hint.** Divide the interval $[a, b]$ into n parts by the points $x_i=aq^i$ $(i=0, 1, 2, \ldots, n)$, where $q=\sqrt[n]{\frac{b}{a}}$. *Ans.* $\frac{b^3-a^3}{3}$.