B. Annex

B.1 Basic Antiderivatives

In all the following formulas, C denotes an arbitrary constant of integration.

1. $\displaystyle\int x^p\,dx = \frac{x^{p+1}}{p+1} + C, \qquad (p \neq -1),$
2. $\displaystyle\int \frac{dx}{x} = \ln|x| + C,$
3. $\displaystyle\int \sin x\,dx = -\cos x + C,$
4. $\displaystyle\int \cos x\,dx = \sin x + C,$
5. $\displaystyle\int \sec^2 x\,dx = \tan x + C,$
6. $\displaystyle\int \csc^2 x\,dx = -\cot x + C,$
7. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C,$
8. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C,$
9. $\displaystyle\int e^x\,dx = e^x + C,$
10. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C, \qquad (a > 0, a \neq 1),$
11. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C,$
12. $\displaystyle\int \frac{dx}{1-x^2} = \operatorname{argtanh} x + C = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + C, \quad |x| < 1,$
13. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C,$
14. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C.$

B.2 Trigonometry

This appendix summarizes the principal trigonometric identities and formulas commonly used in analysis and geometry.

B.2.1 Addition and Subtraction Formulas

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta,\\
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta,\\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.
\end{aligned}$$

B.2.2 Special Cases

$$\begin{aligned}
\cos\left(\frac{\pi}{2}-\alpha\right) &= \sin\alpha, \qquad \sin\left(\frac{\pi}{2}-\alpha\right) = \cos\alpha,\\
\tan\left(\frac{\pi}{2}-\alpha\right) &= \frac{1}{\tan\alpha} = \cot\alpha,\\
\cos(\alpha+n\pi) &= (-1)^n\cos\alpha, \qquad \sin(\alpha+n\pi) = (-1)^n\sin\alpha.
\end{aligned}$$

B.2.3 Double-Angle Formulas

$$\begin{aligned}
\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha,\\
\sin(2\alpha) &= 2\sin\alpha\cos\alpha,\\
\tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha}.
\end{aligned}$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}, \qquad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

B.2.4 Factorization Formulas

$$\begin{aligned}
\cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),\\
\sin\alpha + \sin\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).
\end{aligned}$$

B.2.5 Linearization Formulas

$$\begin{aligned}
\cos\alpha\cos\beta &= \tfrac{1}{2}\big(\cos(\alpha+\beta)+\cos(\alpha-\beta)\big),\\
\sin\alpha\sin\beta &= \tfrac{1}{2}\big(\cos(\alpha-\beta)-\cos(\alpha+\beta)\big),\\
\sin\alpha\cos\beta &= \tfrac{1}{2}\big(\sin(\alpha+\beta)+\sin(\alpha-\beta)\big).
\end{aligned}$$

B.2.6 Rational Parametrization of the Unit Circle

Let $t = \tan\left(\frac{\alpha}{2}\right)$. Then

$$\cos\alpha = \frac{1-t^2}{1+t^2}, \quad \sin\alpha = \frac{2t}{1+t^2}, \quad \tan\alpha = \frac{2t}{1-t^2}.$$

B.3 Comparison of Functions

This appendix presents the fundamental definitions used to compare functions near a limit point a.

Definition B.3.1 — Limit Point. A real number a is said to be a *limit point* of a set $I \subset \mathbb{R}$ if every open interval containing a also contains at least one point of I distinct from a.

Definition B.3.2 — Asymptotic Comparison near a Point. Let $f, g : I \to \mathbb{R}$ be two functions defined on an interval I having a as a limit point.

1. **Equivalence.** When g does not vanish in a neighborhood of a, the relation $f \sim_a g$ holds if and only if

$$\lim_{t \to a} \frac{f(t)}{g(t)} = 1.$$

   In this case, we say that f is *equivalent to* g near a.
2. **Domination.** When g does not vanish in a neighborhood of a, the relation $f =_a O(g)$ holds if and only if $\frac{f(t)}{g(t)}$ remains bounded in a neighborhood of a. In this case, we say that f is *dominated by* g near a.
3. **Negligibility.** When g does not vanish in a neighborhood of a, the relation $f =_a o(g)$ holds if and only if

$$\lim_{t \to a} \frac{f(t)}{g(t)} = 0.$$

   In this case, we say that f is *negligible with respect to* g near a.

Proposition B.3.1 Negligibility and equivalence both imply domination. If $f =_a o(g)$ or $f \sim_a g$, then $f =_a O(g)$.

Warning. The converse is false.

B.4 Common Asymptotic Equivalents

This appendix lists the principal asymptotic equivalents of standard functions near 0. These relations are fundamental tools for studying limits, asymptotic expansions, and the convergence of improper integrals. They also serve as a basis for constructing Taylor series and for comparing the growth rates of functions.

B.4.1 Logarithmic, Exponential, and Power Functions

A polynomial is equivalent near 0 (respectively near $\pm\infty$) to its term of lowest (respectively highest) degree.

$$\ln(1+x) \sim_0 x, \qquad e^x - 1 \sim_0 x, \qquad (1+x)^\alpha - 1 \sim_0 \alpha x.$$

B.4.2 Circular Functions

$$\sin x \sim_0 x, \qquad 1 - \cos x \sim_0 \frac{x^2}{2}, \qquad \tan x \sim_0 x.$$

B.4.3 Inverse Circular Functions

$$\arcsin x \sim_0 x, \qquad \arctan x \sim_0 x,$$

B.4.4 Hyperbolic Functions

$$\sinh x \sim_0 x, \qquad \cosh x - 1 \sim_0 \frac{x^2}{2}, \qquad \tanh x \sim_0 x.$$