1 complex numbers

1.1 Definitions and properties

Definition 2.1

We call a complex number and denote it z, for each ordered pair (x,y) of real numbers.

$\cdot$ the x component is called the real part of z and we denote it as $\operatorname{Re}(z)$

$\cdot$ the y component is called the imaginary part of z and we denote it as $\operatorname{Im}(z)$

$$\cdot\ (x,y) = \left(x^{'},y^{'}\right) \Longleftrightarrow \begin{cases} x = x^{'} \\ \text{and} \\ y = y^{'} \end{cases}$$

$\cdot$ the set of complex number we denote $\mathbb{C}$ and provided with two opirations:

1) Addition $(+)$: $(x,y) + \left(x^{'},y^{'}\right) = \left(x + x^{'}, y + y^{'}\right)$ where $0_{\mathbb{C}} = (0,0)$.

2) Multiplication $(\times)$: $(x,y) \times \left(x^{'},y^{'}\right) = \left(xx^{'} - yy^{'}, xy^{'} + yx^{'}\right)$ where $1_{\mathbb{C}} = (1,0)$.

Remarks

1) the set of complex numbers is a commutative field.

2) the neutral element is $0_{\mathbb{C}} = (0,0)$ we denote it as 0 and the unit element is $1_{\mathbb{C}} = (1,0)$ we denote it as 1.

Notation

1) the complex number $(0,1)$ is noted i.

Theorem 2.1

1) we have $i^2 = -1$.

2) if $x \in \mathbb{R}$ and $y \in \mathbb{R}$, then $(x,y) = x + iy$

Proof

1) $i^2 = (0,1) \times (0,1) = (-1,0) = -(1,0) = -1$

2) we have $(x,y) = (x,0) + (0,1)(0,y) = x + iy$.

Definition 2.2

1) the complex number iy with $y \in \mathbb{R}^*$ is called pure imajinary.

2) the conjugate of a complex number $z = x + iy$ ($x \in \mathbb{R}$ and $y \in \mathbb{R}$) is the complex number $\overline{z}$ where $\overline{z} = x - iy$

Properties

z and w are complex numbers, so

1) $\overline{\overline{z}} = z$

2) $z + \overline{z} = 2\operatorname{Re}(z)$

3) $z + \overline{z} = 2\operatorname{Im}(z)$

4) $\overline{z + w} = \overline{z} + \overline{w}$

5) $\overline{z \times w} = \overline{z} \times \overline{w}$

6) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ $(w \neq 0\ )$

7) $z = \overline{z} \Longleftrightarrow z \in \mathbb{R}$

8) $z = \overline{z} \Longleftrightarrow z$ pure imajinary

Definition 2.3

we call the module of a coplex number z the positive real number $|z|$ where $|z| = \sqrt{z.\overline{z}} = \sqrt{x^2 + y^2}$.

Properties

z and w are complex numbers, so

1) $|\text{Re}(z)| \ \leq \ |z|$; $|\text{Im}(z)| \ \leq \ |z|$ 3) $|z.w| = |z|\,|w|$

2) $|z| = 0 \iff z = 0$ 4) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ $(w \neq 0)$

5) $|z + w| \leq |z| + |w|$ (Triangle inequality)

1.2 The trigonometric form

Definition 2.4

Let $z \in \mathbb{C}^*$; There exists a class of real $\theta + 2\pi k\,(k \in \mathbb{Z})$ (or $\theta\,[2\pi]$) where

$$z = |z|\, e^{i\theta}$$

we notice $Arg\,(z) = \theta\,[2\pi]$; $|z| = \rho$ $(\rho > 0)$,then:

$$z = \underbrace{x + iy}_{\text{Algebraic form}} = \underbrace{\rho e^{i\theta}}_{\text{exponentiel form}} = \underbrace{\rho\,(\cos\theta + \sin\theta)}_{\text{Trigonometric form}}$$

where

$$\rho = \sqrt{x^2 + y^2} \ ; \ \cos\theta = \frac{x}{\rho} \ ; \ \sin\theta = \frac{y}{\rho}.$$

Properties

Let $z = \rho e^{i\theta}$ and $w = re^{i\varphi}$

1) $z.w = \rho r e^{i(\theta+\varphi)}$ 3) $\frac{1}{z} = \frac{1}{\rho} e^{-i\theta}$

2) $\frac{z}{w} = \frac{\rho}{r} e^{i(\theta-\varphi)}$ 4) $\overline{z} = \rho e^{-i\theta}$

5) $z^n = \rho^n e^{i\theta n} = \rho^n\,(\cos n\theta + \sin n\theta)$ (De moivre's formula ($Abraham\ De\ Moivre(1667 - 1754)$))

6) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$; $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ (Euler formula)

1.3 Application of complex numbers to trigonometry

1.3.1 Calculates $\cos nx$ and $\sin nx$ based en $\cos x$ and $\sin x$

we have:

$$\begin{aligned}
\cos nx + i \sin nx &= (\cos x + i \sin x)^n \\
&= \sum_{k=0}^{n} C_n^k \cos^{n-k} x \sin^k x \\
&= C_n^0 \cos^n x - C_n^2 \cos^{n-2} x \sin^2 x + C_n^4 \cos^{n-4} x \sin^4 x + ... \\
&\quad + i \left(C_n^1 \cos^{n-1} x \sin x - C_n^3 \cos^{n-3} x \sin^3 x + C_n^5 \cos^{n-5} x \sin^5 x + ... \right).
\end{aligned}$$

So

$$\begin{cases}
\cos nx = C_n^0 \cos^n x - C_n^2 \cos^{n-2} x \sin^2 x + C_n^4 \cos^{n-4} x \sin^4 x - C_n^6 \cos^{n-6} x \sin^6 x + ... \\
\sin nx = C_n^1 \cos^{n-1} x \sin x - C_n^3 \cos^{n-3} x \sin^3 x + C_n^5 \cos^{n-5} x \sin^5 x + ...
\end{cases}$$

or

$$\begin{aligned}
\cos nx &= \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i C_n^{2i} \cos^{n-2i} x \sin^{2i} x \\
\sin nx &= \sum_{i=1}^{\left[\frac{n+1}{2}\right]} (-1)^{i-1} C_n^{2i-1} \cos^{n-2i+1} x \sin^{2i-1} x
\end{aligned}$$

where $[\frac{n}{2}]$ denotes the integer part of the rational number $\frac{n}{2}$.

1.3.2 Linearization of trigonometric polynomials

For obtain linearization of $\cos^n x$ and $\sin^n x$ we use:

$$\forall k \in \mathbb{Z}; \forall x \in \mathbb{R} : \cos kx = \frac{e^{ikx} + e^{-ikx}}{2}; \sin kx = \frac{e^{ikx} - e^{-ikx}}{2i} \text{ (Euler formula)}$$

Example: write in linear form $\cos^3 x, \sin^4 x$
we have

$$\begin{aligned}
\cos^3 x &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3 \\
&= \frac{1}{8}\left(e^{3ix} + 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} + e^{-3ix}\right) \\
&= \frac{1}{8}\left(e^{3ix} + e^{-3ix} + 3\left(e^{ix} + e^{-ix}\right)\right) \\
&= \frac{1}{8}\left(2\cos 3x + 3\left(2\cos x\right)\right) \\
&= \frac{1}{4}\cos 3x + \frac{3}{4}\cos x.
\end{aligned}$$

$$\begin{aligned}
\sin^4 x &= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^4 \\
&= \frac{1}{16}\left(e^{4ix} - 4e^{3ix}e^{-ix} + 6e^{2ix}e^{-2ix} - 4e^{ix}e^{-3ix} + e^{-4ix}\right) \\
&= \frac{1}{16}\left(e^{4ix} + e^{-4ix} - 4\left(e^{2ix} + e^{-2ix}\right) + 6\right) \\
&= \frac{1}{16}\left(2\cos 4x - 4\left(2\cos 2x\right) + 6\right) \\
&= \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}.
\end{aligned}$$

1.3.3 nth roots of complex number

Definition Let $n \in \mathbb{N}^* - \{1\}$

An nth root of complex number a is a complex number z such that $z^n = a$.

Example

We have $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^2 = \left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)^2 = i$, so $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ and $-\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$ they are square roots of the complex number i

Theorem Let $n \in \mathbb{N}^* - \{1\}$

Any nonzero complex number has exactly n distinct nth roots and if $a = re^{i\theta}$, then the solutions to $z^n = a$ are given by $z_k = \sqrt[n]{r}e^{i\frac{\theta+2\pi k}{n}}$ for $k \in \{0, 1, ..., n-1\}$.

Proof

Suppose thatb $z = \rho e^{i\alpha}$, so

$$\begin{aligned} z^n &= a \Longleftrightarrow \rho^n e^{i\alpha n} = re^{i\theta} \\ &\Longleftrightarrow \rho^n = r \quad \text{and} \quad \alpha n = \theta + 2\pi k, k \in \mathbb{Z}. \end{aligned}$$

The expression for z takes on n different values for $k = 0, 1, ..., n-1$, and the values

start to repeat for $k = n, n+1, ...$

Hence the expression for the n nth roots of a:

$$z_k = \sqrt[n]{r}e^{i\frac{\theta+2\pi k}{n}} \text{ for } k \in \{0, 1, ..., n-1\}$$

Remark

The roots lie on a circle of radius $\sqrt[n]{r}$ centred at the origin and spaced out evenly by angles of $\frac{2\pi}{n}$.

Example

The n nth roots of unity are therefore the numbers

$$z_k = e^{i\frac{2\pi k}{n}} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} \text{ for } k \in \{0, 1, ..., n-1\}$$