# MULTIPLE INTEGRALS

## 2.1 DOUBLE INTEGRALS

In an $xy$-plane we consider a closed* domain $D$ bounded by a line $L$.

In $D$ let there be given a continuous function

$$z = f(x, y)$$

Using arbitrary lines we divide the domain $D$ into $n$ parts

$$\Delta s_1, \Delta s_2, \Delta s_3, \ldots, \Delta s_n$$

(Fig. 43) which we shall call subdomains. So as not to introduce new symbols we will denote by $\Delta s_1, \ldots, \Delta s_n$ both the subdomains and their areas. In each subdomain $\Delta s_i$ (it is immaterial whether in the interior or on the boundary) take a point $P_i$; we will then have $n$ points:

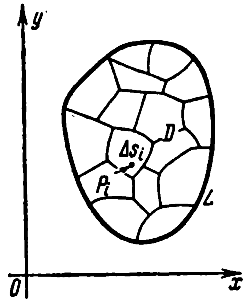

Fig. 43

$$P_1, P_2, \ldots, P_n$$

We denote by $f(P_1), f(P_2), \ldots, f(P_n)$ the values of the functions at the chosen points and then form the sum of the products $f(P_i)\,\Delta s_i$:

$$V_n = f(P_1)\,\Delta s_1 + f(P_2)\,\Delta s_2 + \ldots + f(P_n)\,\Delta s_n = \sum_{i=1}^{n} f(P_i)\,\Delta s_i \qquad (1)$$

This is the *integral sum* of the function $f(x, y)$ in the domain $D$.

If $f \geqslant 0$ in $D$, then each term $f(P_i)\,\Delta s_i$ may be represented geometrically as the volume of a small cylinder with base $\Delta s_i$ and altitude $f(P_i)$.

The sum $V_n$ is the sum of the volumes of the indicated elementary cylinders, that is, the volume of a certain "step-like" solid (Fig. 44).

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* A domain $D$ is called *closed* if it is bounded by a closed line, and the points lying on the boundary are considered as belonging to $D$.

Consider an arbitrary sequence of integral sums formed by means of the function $f(x, y)$ for the given domain $D$,

$$V_{n_1}, V_{n_2}, \ldots, V_{n_k}, \ldots \qquad (2)$$

for different ways of partitioning $D$ into subdomains $\Delta s_i$. We shall assume that the maximum diameter of the subdomains $\Delta s_i$ approaches zero as $n_k \to \infty$. Then the following proposition, which we give without proof, holds true.

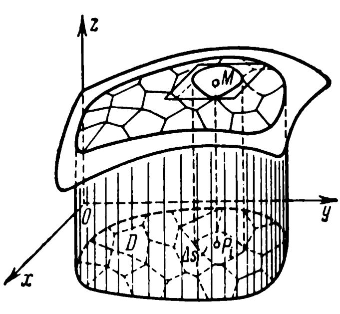


Fig. 44

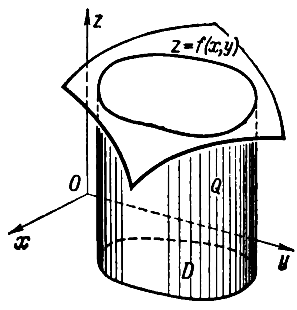


Fig. 45

**Theorem 1.** *If a function $f(x, y)$ is continuous in a closed domain $D$, then the sequence (2) of integral sums (1) has a limit if the maximum diameter of the subdomains $\Delta s_i$ approaches zero as $n_k \to \infty$. This limit is the same for any sequence of type (2), that is, it is independent either of the way $D$ is partitioned into subdomains $\Delta s_i$ or of the choice of the point $P_i$ inside a subdomain $\Delta s_i$.*

This limit is called the *double integral* of the function $f(x, y)$ over $D$ and is denoted by

$$\iint_D f(P)\, ds \quad \text{or} \quad \iint_D f(x, y)\, dx\, dy,$$

that is,

$$\lim_{\text{diam}\, \Delta s_i \to 0} \sum_{i=1}^{n} f(P_i)\, \Delta s_i = \iint_D f(x, y)\, dx\, dy$$

This domain $D$ is called the *domain (region) of integration.*

If $f(x, y) \geqslant 0$, then the double integral of $f(x, y)$ over $D$ is equal to the volume of the solid $Q$ bounded by surface $z = f(x, y)$, the plane $z = 0$, and a cylindrical surface whose generators are parallel to the $z$-axis, while the directrix is the boundary of the domain $D$ (Fig. 45).

Now consider the following theorems about the double integral.

**Theorem 2.** *The double integral of a sum of two functions* $\varphi(x, y)+\psi(x, y)$ *over a domain* $D$ *is equal to the sum of the double integrals over* $D$ *of each of the functions taken separately:*

$$\iint_D [\varphi(x, y)+\psi(x, y)]\,ds = \iint_D \varphi(x, y)\,ds + \iint_D \psi(x, y)\,ds$$

**Theorem 3.** *A constant factor may be taken outside the double integral sign:*
*if* $a=\text{const}$, *then*

$$\iint_D a\varphi(x, y)\,ds = a\iint_D \varphi(x, y)\,ds$$

The proof of both theorems is exactly the same as that of the corresponding theorems for the definite integral (see. Sec. 11.3, Vol. I).

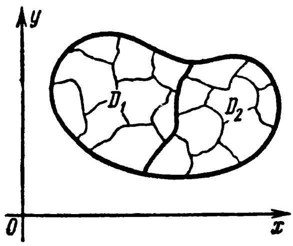


Fig. 46

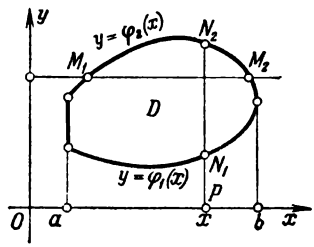


Fig. 47

**Theorem 4.** *If a region* $D$ *is divided into two domains* $D_1$ *and* $D_2$ *without common interior points, and a function* $f(x, y)$ *is continuous at all points of* $D$, *then*

$$\iint_D f(x, y)\,dx\,dy = \iint_{D_1} f(x, y)\,dx\,dy + \iint_{D_2} f(x, y)\,dx\,dy \qquad (3)$$

**Proof.** The integral sum over $D$ may be given in the form (Fig. 46)

$$\sum_D f(P_i)\,\Delta s_i = \sum_{D_1} f(P_i)\,\Delta s_i + \sum_{D_2} f(P_i)\,\Delta s_i \qquad (4)$$

where the first sum contains terms that correspond to the subdomains of $D_1$, the second, those corresponding to the subdomains of $D_2$. Indeed, since the double integral does not depend on the manner of partition, we divide $D$ so that the common boundary of the domains $D_1$ and $D_2$ is a boundary of the subdomains $\Delta s_i$. Passing to the limit in (4) as $\Delta s_i \to 0$, we get (3). This theorem is obviously true for any number of terms.

## 2.2 CALCULATING DOUBLE INTEGRALS

Let a domain $D$ lying in the $xy$-plane be such that any straight line parallel to one of the coordinate axes (for example, the $y$-axis) and passing through an **interior*** point of the domain, cuts the boundary of the domain at two points $N_1$ and $N_2$ (Fig. 47).

In this case we assume that the domain $D$ is bounded by the lines: $y=\varphi_1(x)$, $y=\varphi_2(x)$, $x=a$, $x=b$ and that

$$\varphi_1(x)\leqslant\varphi_2(x),\quad a<b$$

while the functions $\varphi_1(x)$ and $\varphi_2(x)$ are continuous on the interval $[a, b]$. We shall call such a domain *regular in the y-direction*. The definition is similar for a domain *regular in the x-direction*.

A domain that is regular in both $x$- and $y$-directions we shall simply call a *regular domain*. In Fig. 47 we have a regular domain $D$.

Let the function $f(x, y)$ be continuous in $D$.

Consider the expression

$$I_D=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy\right)dx$$

which we shall call a *twofold iterated* integral of $f(x, y)$ over $D$. In this expression we first calculate the integral in the parentheses (the integration is performed with respect to $y$ while $x$ is considered to be constant). The integration yields a continuous** function of $x$:

$$\Phi(x)=\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy$$

Integrating this function with respect to $x$ from $a$ to $b$,

$$I_D=\int_a^b\Phi(x)\,dx$$

we get a certain constant.

**Example.** Calculate the twofold iterated integral

$$I_D=\int_0^1\left(\int_0^{x^2}(x^2+y^2)\,dy\right)dx$$

**Solution.** First calculate the inner integral (in brackets):

$$\Phi(x)=\int_0^{x^2}(x^2+y^2)\,dy=\left(x^2y+\frac{y^3}{3}\right)_0^{x^2}=x^2x^2+\frac{(x^2)^3}{3}=x^4+\frac{x^6}{3}$$

* An interior point of a domain is one that does not lie on its boundary.

** We do not prove here that the function $\Phi(x)$ is continuous.

Integrating the function obtained from 0 to 1, we find

$$\int_0^1 \left(x^4+\frac{x^6}{3}\right)dx=\left(\frac{x^5}{5}+\frac{x^7}{3\cdot 7}\right)_0^1=\frac{1}{5}+\frac{1}{21}=\frac{26}{105}$$

Determine the domain $D$. Here, $D$ is the domain bounded by the lines (Fig. 48)

$$y=0, \quad x=0, \quad y=x^2, \quad x=1$$

It may happen that the domain $D$ is such that one of the functions $y=\varphi_1(x)$, $y=\varphi_2(x)$ cannot be represented by one analytic

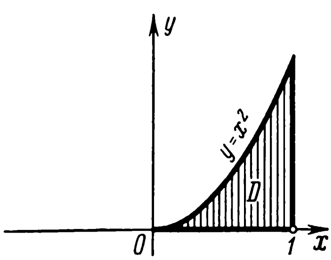


Fig. 48

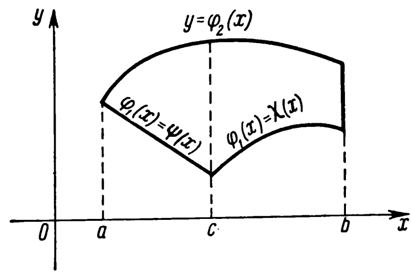


Fig. 49

expression over the entire range of $x$ (from $x=a$ to $x=b$). For example, let $a<c<b$, and

$$\varphi_1(x)=\psi(x) \text{ on the interval } [a,\ c]$$

$$\varphi_1(x)=\chi(x) \text{ on the interval } [c,\ b]$$

where $\psi(x)$ and $\chi(x)$ are analytically specified functions (Fig. 49). Then the twofold iterated integral will be written as follows:

$$\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy\right)dx$$

$$=\int_a^c\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy\right)dx+\int_c^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy\right)dx$$

$$=\int_a^c\left(\int_{\psi(x)}^{\varphi_2(x)} f(x,\ y)\,dy\right)dx+\int_c^b\left(\int_{\chi(x)}^{\varphi_2(x)} f(x,\ y)\,dy\right)dx$$

The first of these equations is written on the basis of a familiar property of the definite integral, the second, by virtue of the fact that on the interval $[a,\ c]$ we have $\varphi_1(x)\equiv\psi(x)$, and on the interval $[c,\ b]$ we have $\varphi_1(x)\equiv\chi(x)$.

We would also have a similar notation for the twofold iterated integral if the function $\varphi_2(x)$ were defined by different analytic expressions on different subintervals of the interval $[a,\ b]$.

Let us establish some properties of a twofold iterated integral.

**Property 1.** *If a regular y-direction domain D is divided into two domains $D_1$ and $D_2$ by a straight line parallel to the y-axis or the x-axis, then the twofold iterated integral $I_D$ over D will be equal to the sum of such integrals over $D_1$ and $D_2$; that is,*

$$I_D = I_{D_1} + I_{D_2} \tag{1}$$

**Proof.** (a) Let the straight line $x = c$ $(a < c < b)$ divide the region $D$ into two regular $y$-direction domains* $D_1$ and $D_2$. Then

$$I_D = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy \right) dx = \int_a^b \Phi(x)\,dx = \int_a^c \Phi(x)\,dx + \int_c^b \Phi(x)\,dx$$

$$= \int_a^c \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy \right) dx + \int_c^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy \right) dx = I_{D_1} + I_{D_2}$$

(b) Let the straight line $y = h$ divide the domain $D$ into two regular $y$-direction domains $D_1$ and $D_2$ as shown in Fig. 50. Denote by $M_1$ and $M_2$ the points of intersection of the straight line $y = h$ with the boundary $L$ of $D$. Denote the abscissas of these points by $a_1$ and $b_1$.

The domain $D_1$ is bounded by continuous lines:

(1) $y = \varphi_1(x)$;

(2) the curve $A_1M_1M_2B$, whose equation we shall conditionally write in the form

$$y = \varphi_1^*(x)$$

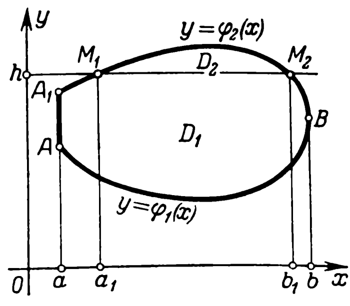


Fig. 50

having in view that $\varphi_1^*(x) = \varphi_2(x)$ when $a \leqslant x \leqslant a_1$ and when $b_1 \leqslant x \leqslant b$ and that

$$\varphi_1^*(x) = h \quad \text{when } a_1 \leqslant x \leqslant b_1;$$

(3) by the straight lines $x = a$, $x = b$.

The domain $D_2$ is bounded by the lines

$$y = \varphi_1^*(x), \quad y = \varphi_2(x), \quad \text{where } a_1 \leqslant x \leqslant b_1$$

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* The fact that a part of the boundary of the domain $D_1$ (and of $D_2$) is a portion of the vertical straight line does not stop this domain from being regular in the $y$-direction: for a domain to be regular, it is only necessary that any vertical straight line passing through an interior point of the domain should have no more than two common points with the boundary.

We write the identity by applying to the inner integral the theorem on partitioning the interval of integration:

$$I_D=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy\right)dx$$

$$=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_1^*(x)} f(x, y)\,dy+\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy\right)dx$$

$$=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_1^*(x)} f(x, y)\,dy\right)dx+\int_a^b\left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy\right)dx$$

We break up the latter integral into three integrals and apply to the outer integral the theorem on partitioning the interval of integration:

$$\int_a^b\left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy\right)dx=\int_a^{a_1}\left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy\right)dx$$

$$+\int_{a_1}^{b_1}\left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy\right)dx+\int_{b_1}^{b}\left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy\right)dx$$

since $\varphi_1^*(x)=\varphi_2(x)$ on $[a, a_1]$ and on $[b_1, b]$, it follows that the first and third integrals are identically zero. Therefore,

$$I_D=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_1^*(x)} f(x, y)\,dy\right)dx+\int_{a_1}^{b_1}\left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy\right)dx$$

Here, the first integral is a twofold iterated integral over $D_1$, the second, over $D_2$. Consequently,

$$I_D=I_{D_1}+I_{D_2}$$

The proof will be similar for any position of the cutting straight line $M_1M_2$. If $M_1M_2$ divides $D$ into three or a larger number of domains, we get a relation similar to (1), in the first part of which we will have the appropriate number of terms.

**Corollary.** We can again divide each of the domains obtained (using a straight line parallel to the $y$-axis or $x$-axis) into regular $y$-direction domains, and we can apply to them equation (1).

Thus, $D$ may be divided by straight lines parallel to the coordinate axes into any number of regular domains

$$D_1, D_2, D_3, \ldots, D_i$$

and the assertion that the *twofold iterated integral over D is equal to the sum of twofold iterated integrals over the subdomains* holds; that is (Fig. 51),

$$I_D = I_{D_1} + I_{D_2} + I_{D_3} + \ldots + I_{D_i} \quad (2)$$

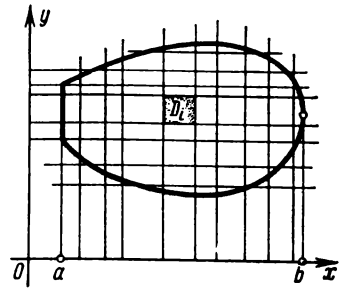


Fig. 51

**Property 2 (Evaluation of an iterated integral).** *Let m and M be the least and greatest values of the function f(x, y) in the domain D. Denote by S the area of D. Then we have the relation*

$$mS \leqslant \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \right) dx \leqslant MS \qquad (3)$$

**Proof.** Evaluate the inner integral denoting it by $\Phi(x)$:

$$\Phi(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \leqslant \int_{\varphi_1(x)}^{\varphi_2(x)} M\, dy = M[\varphi_2(x) - \varphi_1(x)]$$

We then have

$$I_D = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \right) dx \leqslant \int_a^b M[\varphi_2(x) - \varphi_1(x)]\, dx = MS$$

that is,

$$I_D \leqslant MS \qquad (3')$$

Similarly

$$\Phi(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \geqslant \int_{\varphi_1(x)}^{\varphi_2(x)} m\, dy = m[\varphi_2(x) - \varphi_1(x)]$$

$$I_D = \int_a^b \Phi(x)\, dx \geqslant \int_a^b m[\varphi_2(x) - \varphi_1(x)]\, dx = mS$$

that is,

$$I_D \geqslant mS \qquad (3'')$$

From the inequalities (3′) and (3″) follows the relation (3):

$$mS \leqslant I_D \leqslant MS$$

In the next section we will determine the geometric meaning of this theorem.

**Property 3 (Mean-value theorem).** *A twofold iterated integral $I_D$ of a continuous function $f(x, y)$ over a domain $D$ with area $S$ is equal to the product of the area $S$ by the value of the function at some point $P$ in $D$; that is,*

$$\int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy \right) dx = f(P)\,S \tag{4}$$

**Proof.** From (3) we obtain

$$m \leqslant \frac{1}{S} I_D \leqslant M$$

The number $\frac{1}{S} I_D$ lies between the greatest and least values of $f(x, y)$ in $D$. Due to the continuity of the function $f(x, y)$, at some point $P$ of $D$ it takes on a value equal to the number $\frac{1}{S} I_D$; that is,.

$$\frac{1}{S} I_D = f(P)$$

whence

$$I_D = f(P)\,S \tag{5}$$

### 2.3 CALCULATING DOUBLE INTEGRALS (CONTINUED)

**Theorem.** *The double integral of a continuous function $f(x, y)$ over a regular domain $D$ is equal to the twofold iterated integral of this function over $D$; that is,* *

$$\iint_D f(x, y)\,dx\,dy = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy \right) dx$$

**Proof.** Partition the domain $D$ with straight lines parallel to the coordinate axes into $n$ regular (rectangular) subdomains:

$$\Delta s_1,\ \Delta s_2,\ \ldots,\ \Delta s_n$$

By Property 1 [formula (2)] of the preceding section we have

$$I_D = I_{\Delta s_1} + I_{\Delta s_2} + \ldots + I_{\Delta s_n} = \sum_{i=1}^{n} I_{\Delta s_i} \tag{1}$$

We transform each of the terms on the right by the mean-value theorem for a twofold iterated integral:

$$I_{\Delta s_i} = f(P_i)\,\Delta s_i$$

* Here, we again assume that the domain $D$ is regular in the $y$-direction and bounded by the lines $y = \varphi_1(x)$, $y = \varphi_2(x)$, $x = a$, $x = b$.

Then (1) takes the form

$$I_D = f(P_1)\,\Delta s_1 + f(P_2)\,\Delta s_2 + \ldots + f(P_n)\,\Delta s_n = \sum_{i=1}^{n} f(P_i)\,\Delta s_i \qquad (2)$$

where $P_i$ is some point of the subdomain $\Delta s_i$. On the right is the integral sum of the function $f(x, y)$ over $D$. From the existence theorem of a double integral it follows that the limit of this sum, as $n \longrightarrow \infty$ and as the greatest diameter of the subdomains $\Delta s_i$ approaches zero, exists and is equal to the double integral of $f(x, y)$ over $D$. The value of the double integral $I_D$ on the left side of (2) does not depend on $n$. Thus, passing to the limit in (2), we obtain

$$I_D = \lim_{\operatorname{diam} \Delta s_i \to 0} \sum f(P_i)\,\Delta s_i = \iint_D f(x, y)\,dx\,dy$$

or

$$\iint_D f(x, y)\,dx\,dy = I_D \qquad (3)$$

Writing out in full the expression of the twofold iterated integral $I_D$, we finally get

$$\iint_D f(x, y)\,dx\,dy = \int_a^b \left[ \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy \right] dx \qquad (4)$$

**Note 1.** For the case where $f(x, y) \geqslant 0$, formula (4) has a pictorial geometric interpretation. Consider a solid bounded by a surface $z = f(x, y)$, a plane $z = 0$, and a cylindrical surface whose generators are parallel to the $z$-axis and the directrix of which is the boundary of the region $D$ (Fig. 52). Calculate the volume of this solid $V$. It has already been shown that the volume of this solid is equal to the double integral of the function $f(x, y)$ over the domain $D$:

$$V = \iint_D f(x, y)\,dx\,dy \qquad (5)$$

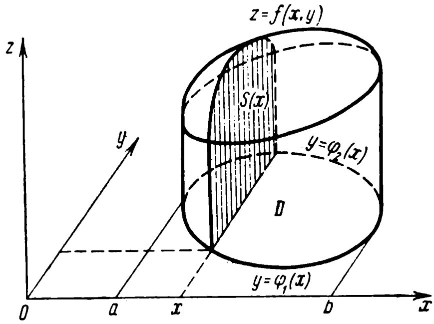

Fig. 52

Now let us calculate the volume of this solid using the results of Sec. 12.4, Vol. I, on the evaluation of the volume of a solid from the areas of parallel sections (slices). Draw the plane $x = \text{const}$ $(a < x < b)$ that cuts the solid. Calculate the area $S(x)$ of the figure obtained in the section $x = \text{const}$. This figure is a curvilinear trapezoid bounded by the lines $z = f(x, y)$ $(x = \text{const})$, $z = 0$, $y = \varphi_1(x)$, $y = \varphi_2(x)$. Hence, the area can be expressed by the integral

$$S(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \tag{6}$$

Knowing the areas of parallel sections, it is easy to find the volume of the solid:

$$V = \int_a^b S(x)\, dx$$

or, substituting expression (6) for the area $S(x)$, we get

$$V = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \right) dx \tag{7}$$

In formulas (5) and (7) the left sides are equal; and so the right sides are equal too:

$$\iint_D f(x, y)\, dx\, dy = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \right) dx$$

It is now easy to figure out the geometric meaning of theorem on evaluating a twofold iterated integral (Property 2, Sec. 2.2): the volume $V$ of a solid bounded by the surface $z = f(x, y)$, the plane $z = 0$, and a cylindrical surface whose directrix is the boun-

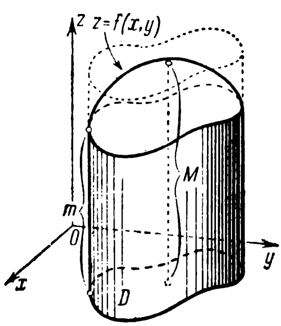


Fig. 53

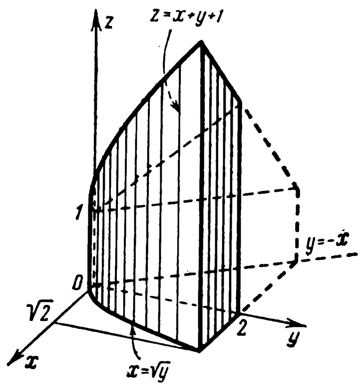


Fig. 54

dary of the region $D$, exceeds the volume of a cylinder with base area $S$ and altitude $m$, but is less than the volume of a cylinder with base area $S$ and altitude $M$ [where $m$ and $M$ are the least and greatest values of the function $z = f(x, y)$ in the domain $D$ (Fig. 53)]. This follows from the fact that the twofold iterated integral $I_D$ is equal to the volume $V$ of this solid.

**Example 1.** Evaluate the double integral $\iint_D (4-x^2-y^2)\,dx\,dy$ if the domain $D$ is bounded by the straight lines $x=0$, $x=1$, $y=0$, and $y=\frac{3}{2}$.

**Solution.** By the formula

$$V=\int_0^{3/2}\left[\int_0^1 (4-x^2-y^2)\,dx\right]dy=\int_0^{3/2}\left[4x-y^2x-\frac{x^3}{3}\right]_0^1 dy$$

$$=\int_0^{3/2}\left(4-y^2-\frac{1}{3}\right)dy=\left(4y-\frac{y^3}{3}-\frac{1}{3}y\right)\Big|_0^{3/2}=\frac{35}{8}$$

**Example 2.** Evaluate the double integral of the function $f(x, y)=1+x+y$ over a region bounded by the lines $y=-x$, $x=\sqrt{y}$, $y=2$, $z=0$ (Fig. 54).

**Solution.**

$$V=\int_0^2\left[\int_{-y}^{\sqrt{y}} (1+x+y)\,dx\right]dy=\int_0^2\left[x+xy+\frac{x^2}{2}\right]_{-y}^{\sqrt{y}} dy$$

$$=\int_0^2\left[\left(\sqrt{y}+y\sqrt{y}+\frac{y}{2}\right)-\left(-y-y^2+\frac{y^2}{2}\right)\right]dy$$

$$=\int_0^2\left[\sqrt{y}+\frac{3y}{2}+y\sqrt{y}+\frac{y^2}{2}\right]dy$$

$$=\left[\frac{2y^{\frac{3}{2}}}{3}+\frac{3y^2}{4}+\frac{2y^{\frac{5}{2}}}{5}+\frac{y^3}{6}\right]_0^2=\frac{44}{15}\sqrt{2}+\frac{13}{3}$$

**Note 2.** Let a regular $x$-direction domain $D$ be bounded by the lines

$$x=\psi_1(y),\quad x=\psi_2(y),\quad y=c,\quad y=d$$

and let $\psi_1(y)\leqslant\psi_2(y)$ (Fig. 55).

In this case, obviously,

$$\iint_D f(x, y)\,dx\,dy=\int_c^d\left(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\,dx\right)dy \tag{8}$$

To evaluate the double integral we must represent it as a twofold iterated integral. As we have already seen, this may be done in two different ways: either by formula (4) or by formula (8). Depending upon the type of domain $D$ or the integrand in each specific case, we choose one of the formulas to calculate the double integral.

**Example 3.** Change the order of integration in the integral

$$I=\int_0^1\left(\int_x^{\sqrt{x}} f(x, y)\,dy\right)dx$$

**Solution.** The domain of integration is bounded by the straight line $y=x$ and the parabola $y=\sqrt{x}$ (Fig. 56).

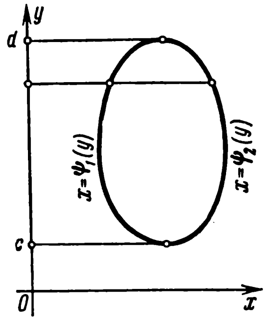


Fig. 55

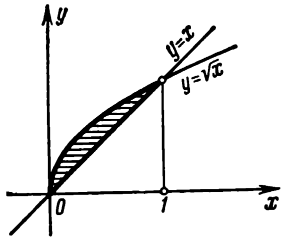


Fig. 56

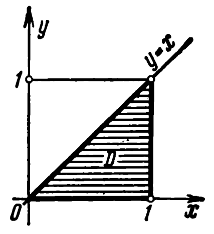


Fig. 57

Every straight line parallel to the $x$-axis cuts the boundary of the domain at no more than two points; hence, we can compute the integral by formula (8), setting

$$\psi_1(y)=y^2, \quad \psi_2(y)=y, \quad 0\leqslant y\leqslant 1$$

then

$$I=\int_0^1\left(\int_{y^2}^{y} f(x, y)\,dx\right)dy$$

**Example 4.** Evaluate $\iint\limits_D e^{\frac{y}{x}}\,ds$ if the domain $D$ is a triangle bounded by the straight lines $y=x$, $y=0$, and $x=1$ (Fig. 57).

**Solution.** Replace this double integral by a twofold iterated integral using formula (4). [If we used formula (8), we would have to integrate the function $e^{\frac{y}{x}}$ with respect to $x$; but this integral is not expressible in terms of elementary functions]:

$$\iint\limits_D e^{\frac{y}{x}}\,ds=\int_0^1\left[\int_0^x e^{\frac{y}{x}}\,dy\right]dx=\int_0^1\left[xe^{\frac{y}{x}}\right]_0^x dx$$

$$=\int_0^1 x(e-1)\,dx=(e-1)\frac{x^2}{2}\bigg|_0^1=\frac{e-1}{2}=0.859\ldots$$

**Note 3.** If the domain $D$ is not regular either in the $x$-direction or the $y$-direction (that is, there exist vertical and horizontal straight lines which, while passing through interior points of the domain, cut the boundary of the domain at more than two points), then we cannot represent the double integral over this domain in the form of a twofold iterated integral. If we manage to partition the irregular domain $D$ into a finite number of regular $x$-direction or $y$-direction domains $D_1, D_2, \ldots, D_n$, then, by evaluating the double integral over each of these subdomains by means of the twofold iterated integral and adding the results obtained, we get the sought-for integral over $D$.

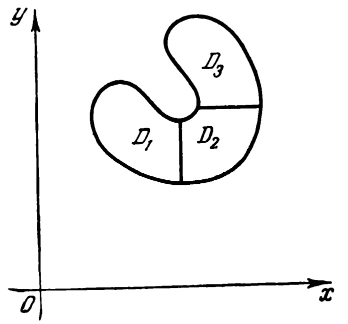


Fig. 58

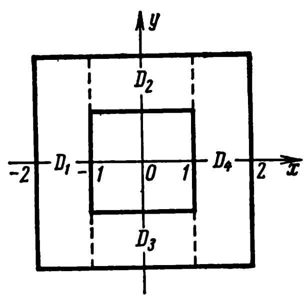


Fig. 59

Fig. 58 is an example of how an irregular domain $D$ may be divided into three regular subdomains $D_1$, $D_2$ and $D_3$.

**Example 5.** Evaluate the double integral

$$\iint_D e^{x+y}\,ds$$

over a domain $D$ which lies between two squares with centre at the origin and with sides parallel to the axes of coordinates, if each side of the inner square is equal to 2 and that of the outer square is 4 (Fig. 59).

**Solution.** $D$ is irregular. However, the straight lines $x=-1$ and $x=1$ divide it into four regular subdomains $D_1$, $D_2$, $D_3$, $D_4$. Therefore,

$$\iint_D e^{x+y}\,ds=\iint_{D_1} e^{x+y}\,ds+\iint_{D_2} e^{x+y}\,ds+\iint_{D_3} e^{x+y}\,ds+\iint_{D_4} e^{x+y}\,ds$$

Representing each of these integrals in the form of a twofold iterated integral, we find

$$\iint_D e^{x+y}\,ds=\int_{-2}^{-1}\left[\int_{-2}^{2} e^{x+y}\,dy\right]dx+\int_{-1}^{1}\left[\int_{1}^{2} e^{x+y}\,dy\right]dx$$

$$+\int_{-1}^{1}\left[\int_{-2}^{-1} e^{x+y}\,dy\right]dx+\int_{1}^{2}\left[\int_{-2}^{2} e^{x+y}\,dy\right]dx$$

$$=(e^2-e^{-2})(e^{-1}-e^{-2})+(e^2-e)(e-e^{-1})+(e^{-1}-e^{-2})(e-e^{-1})$$

$$+(e^2-e^{-2})(e^2-e)=(e^3-e^{-3})(e-e^{-1})=4\sinh 3\sinh 1$$

**Note 4.** From now on, when writing the twofold iterated integral

$$I_D=\int_a^l\left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy\right]dx$$

we will drop the brackets containing the inner integral and will write

$$I_D=\int_a^b\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy\,dx$$

Here, just as in the case when we have brackets, we will consider that the first integration is performed with respect to the variable whose differential is written first, and then with respect to the variable whose differential is written second. [We note, however, that this is not the generally accepted practice; in some books the reverse is done: integration is performed first with respect to the variable whose differential is last.*]

## 2.4 CALCULATING AREAS AND VOLUMES BY MEANS OF DOUBLE INTEGRALS

1. **Volume.** As we saw in Sec. 2.1, the volume $V$ of a solid bounded by a surface $z=f(x,\ y)$, where $f(x,\ y)$ is a nonnegative function, by a plane $z=0$ and by a cylindrical surface whose directrix is the boundary of the domain $D$ and the generators are parallel to the $z$-axis, is equal to the double integral of the function $f(x,\ y)$ over $D$:

$$V=\iint_D f(x,\ y)\,ds$$

**Example 1.** Calculate the volume of a solid bounded by the surfaces $x=0$, $y=0$, $x+y+z=1$, $z=0$ (Fig. 60).

**Solution.**

$$V=\iint_D (1-x-y)\,dy\,dx$$

where $D$ is (in Fig. 60) the hatched triangular region in the $xy$-plane bounded by the straight lines $x=0$, $y=0$, and $x+y=1$. Putting the limits on the double

---

* The following notation is also sometimes used:

$$I_D=\int_a^b\left[\int_{\varphi_1}^{\varphi_2} f(x,\ y)\,dy\right]dx=\int_a^b dx\int_{\varphi_1}^{\varphi_2} f(x,\ y)\,dy$$

integral, we calculate the volume:

$$V=\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx=\int_0^1\left[(1-x)\,y-\frac{y^2}{2}\right]_0^{1-x}dx=\int_0^1\frac{1}{2}(1-x)^2\,dx=\frac{1}{6}$$

Thus, $V=\frac{1}{6}$ cubic units.

**Note 1.** If a solid, the volume of which is being sought, is bounded above by the surface $z=\Phi_2(x, y)\geqslant 0$, and below by the surface $z=\Phi_1(x, y)\geqslant 0$, and the domain $D$ is the projection of

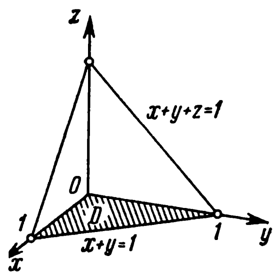


Fig. 60

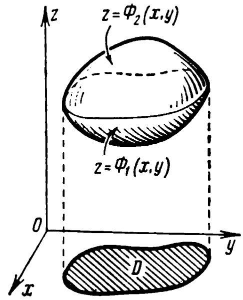


Fig. 61

both surfaces on the $xy$-plane, then the volume $V$ of this solid is equal to the difference between the volumes of the two "cylindrical" bodies; the first of these cylindrical bodies has the domain $D$ for its lower base, and the surface $z=\Phi_2(x, y)$ for its upper base; the second body also has $D$ for its lower base, and the surface $z=\Phi_1(x, y)$ for its upper base (Fig. 61).

Therefore, the volume $V$ is equal to the difference between the two double integrals

$$V=\iint_D \Phi_2(x, y)\,ds-\iint_D \Phi_1(x, y)\,ds$$

or

$$V=\iint_D [\Phi_2(x, y)-\Phi_1(x, y)]\,ds \qquad (1)$$

Further, it is easy to prove that formula (1) holds true not only for the case where $\Phi_1(x, y)$ and $\Phi_2(x, y)$ are nonnegative, but also where $\Phi_1(x, y)$ and $\Phi_2(x, y)$ are any continuous functions that satisfy the relationship

$$\Phi_2(x, y)\geqslant\Phi_1(x, y)$$

**Note 2.** If in the domain $D$ the function $f(x, y)$ changes sign, then we divide the domain into two parts: (1) the subdomain $D_1$

where $f(x, y) \geqslant 0$; (2) the subdomain $D_2$ where $f(x, y) \leqslant 0$. Suppose the subdomains $D_1$ and $D_2$ are such that the double integrals over them exist. Then the integral over $D_1$ will be positive and equal to the volume of the solid lying above the $xy$-plane. The integral over $D_2$ will be negative and equal, in absolute value, to the volume of the solid lying below the $xy$-plane. Thus, the integral over $D$ will be expressed as the difference between the corresponding volumes.

2. **Calculating the area of a plane region.** If we form the integral sum of the function $f(x, y) \equiv 1$ over the domain $D$, then this sum will be equal to the area $S$,

$$S = \sum_{i=1}^{n} 1 \cdot \Delta s_i$$

for any mode of partition. Passing to the limit on the right side of the equation, we get

$$S = \iint_D dx\,dy$$

If $D$ is regular (see, for instance, Fig. 47), then the area will be expressed by the iterated integral

$$S = \int_a^b \left[ \int_{\varphi_1(x)}^{\varphi_2(x)} dy \right] dx$$

Performing the integration in the brackets, we obviously have

$$S = \int_a^b [\varphi_2(x) - \varphi_1(x)]\,dx$$

(cf. Sec. 12.1, Vol. I).

**Example 2.** Calculate the area of a region bounded by the curves

$$y = 2 - x^2, \quad y = x$$

**Solution.** Determine the points of intersection of the given curves (Fig. 62). At the point of intersection the ordinates are equal; that is,

$$x = 2 - x^2$$

whence

$$x^2 + x - 2 = 0$$
$$x_1 = -2$$
$$x_2 = 1$$

We get two points of intersection: $M_1(-2, -2)$, $M_2(1, 1)$. Hence, the required area is

$$S = \int_{-2}^{1} \left( \int_{x}^{2-x^2} dy \right) dx = \int_{-2}^{1} (2 - x^2 - x)\,dx = \left[ 2x - \frac{x^3}{3} - \frac{x^2}{2} \right]_{-2}^{1} = \frac{9}{2}$$

## 2.5 THE DOUBLE INTEGRAL IN POLAR COORDINATES

Suppose that in a polar coordinate system $\theta$, $\rho$, a domain $D$ is given such that each ray* passing through an interior point of the region cuts the boundary of $D$ at no more than two points. Suppose that $D$ is bounded by the curves $\rho=\Phi_1(\theta)$, $\rho=\Phi_2(\theta)$ and the rays $\theta=\alpha$ and $\theta=\beta$, where $\Phi_1(\theta)\leqslant\Phi_2(\theta)$ and $\alpha<\beta$ (Fig. 63). Again we shall call such a region a *regular* domain.

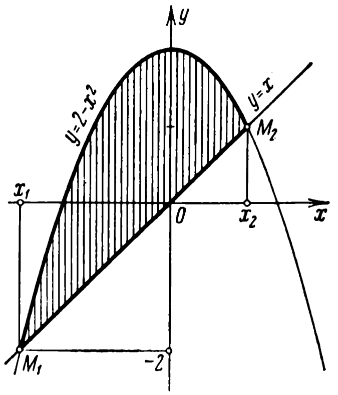


Fig. 62

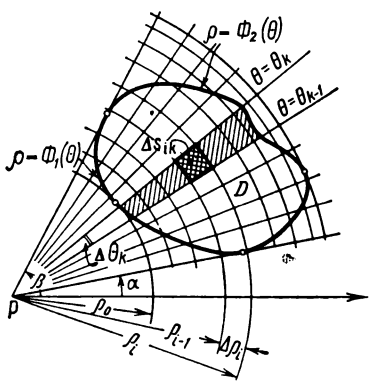


Fig. 63

Let there be given in $D$ a continuous function of the coordinates $\theta$ and $\rho$:

$$z=F(\theta, \rho)$$

We divide $D$ in some way into subdomains $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. Form the (integral) sum

$$V_n=\sum_{k=1}^{n} F(P_k)\,\Delta s_k \tag{1}$$

where $P_k$ is some point in the subdomain $\Delta s_k$.

From the existence theorem of a double integral it follows that as the greatest diameter of the subdomain $\Delta s_k$ approaches zero, there exists a limit $V$ of the integral sum (1). By definition, this limit $V$ is the double integral of the function $F(\theta, \rho)$ over the domain $D$:

$$V=\iint_D F(\theta, \rho)\,ds \tag{2}$$

Let us now evaluate this double integral.

---

* A *ray* is any half-line issuing from the coordinate origin, that is, from the pole $P$.

Since the limit of the sum is independent of the manner of partitioning $D$ into subdomains $\Delta s_k$, we can divide the domain in a way that is most convenient. This most convenient (for purposes of calculation) manner will be to partition the domain by means of the rays $\theta=\theta_0$, $\theta=\theta_1$, $\theta=\theta_2$, ..., $\theta=\theta_n$ (where $\theta_0=\alpha$, $\theta_n=\beta$, $\theta_0<\theta_1<\theta_2<\ldots<\theta_n$) and the concentric circles $\rho=\rho_0$, $\rho=\rho_1$, ..., $\rho=\rho_m$ [where $\rho_0$ is equal to the least value of the function $\Phi_1(\theta)$, and $\rho_m$, to the greatest value of the function $\Phi_2(\theta)$ in the interval $\alpha\leqslant\theta\leqslant\beta$, $\rho_0<\rho_1<\ldots<\rho_m$].

Denote by $\Delta s_{ik}$ the subdomain bounded by the lines $\rho=\rho_{i-1}$, $\rho=\rho_i$, $\theta=\theta_{k-1}$, $\theta=\theta_k$.

The subdomains $\Delta s_{ik}$ will be of three kinds:

(1) those that are not cut by the boundary and lie in $D$;
(2) those that are not cut by the boundary and lie outside $D$;
(3) those that are cut by the boundary of $D$.

The sum of the terms corresponding to the cut subdomains have zero as their limit when $\Delta\theta_k\to 0$ and $\Delta\rho_i\to 0$ and for this reason these terms will be disregarded. The subdomains $\Delta s_{ik}$ that lie outside $D$ do not interest us since they do not enter into the sum. Thus, the sum may be written as follows:

$$V_n=\sum_{k=1}^{n}\left[\sum_i F(P_{ik})\,\Delta s_{ik}\right]$$

where $P_{ik}$ is an arbitrary point of the subdomain $\Delta s_{ik}$.

The double summation sign here should be understood as meaning that we first perform the summation with respect to the index $i$, holding $k$ fast (that is, we pick out all terms that correspond to the subdomains lying between two adjacent rays*). The outer summation sign signifies that we take together all the sums obtained in the first summation (that is, we sum with respect to the index $k$).

Let us find the expression of the area of the subdomain $\Delta s_{ik}$ that is not cut by the boundary of the domain. It will be equal to the difference of the areas of the two sectors:

$$\Delta s_{ik}=\frac{1}{2}(\rho_i+\Delta\rho_i)^2\,\Delta\theta_k-\frac{1}{2}\rho_i^2\Delta\theta_k=\left(\rho_i+\frac{\Delta\rho_i}{2}\right)\Delta\rho_i\,\Delta\theta_k$$

or

$$\Delta s_{ik}=\rho_i^*\,\Delta\rho_i\Delta\theta_k,\quad \text{where } \rho_i<\rho_i^*<\rho_i+\Delta\rho_i$$

* Note that in summing with respect to the index $i$ this index will not run through all values from 1 to $m$, because not all of the subdomains lying between the rays $\theta=\theta_k$ and $\theta=\theta_{k+1}$ belong to $D$.

Thus, the integral sum will have the form *

$$V_n = \sum_{k=1}^{n} \left[ \sum_i F(\theta_k^*, \rho_i^*) \rho_i^* \Delta\rho_i \Delta\theta_k \right]$$

where $P(\theta_k^*, \rho_i^*)$ is a point of the subdomain $\Delta s_{ik}$. Now take the factor $\Delta\theta_k$ outside the sign of the inner sum (this is permissible since it is a common factor for all the terms of this sum):

$$V_n = \sum_{k=1}^{n} \left[ \sum_i F(\theta_k^*, \rho_i^*) \rho_i^* \Delta\rho_i \right] \Delta\theta_k$$

Suppose that $\Delta\rho_i \longrightarrow 0$ and $\Delta\theta_k$ remains constant. Then the expression in the brackets will tend to the integral

$$\int_{\Phi_1(\theta_k^*)}^{\Phi_2(\theta_k^*)} F(\theta_k^*, \rho) \rho \, d\rho$$

Now, assuming that $\Delta\theta_k \longrightarrow 0$, we finally get **

$$V = \int_{\alpha}^{\beta} \left( \int_{\Phi_1(\theta)}^{\Phi_2(\theta)} F(\theta, \rho) \rho \, d\rho \right) d\theta \qquad (3)$$

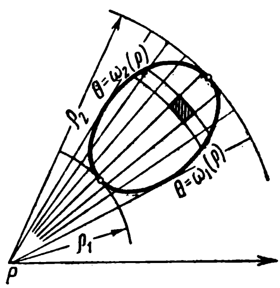

Fig. 64

Formula (3) is used to compute double integrals in polar coordinates.

If the first integration is performed with respect to $\theta$ and the second one to $\rho$, then we get the formula (Fig. 64)

$$V = \int_{\rho_1}^{\rho_2} \left( \int_{\omega_1(\rho)}^{\omega_2(\rho)} F(\theta, \rho) \, d\theta \right) \rho \, d\rho \qquad (3')$$

* We can consider the integral sum in this form because the limit of the sum does not depend on the position of the point inside the subdomain.

** Our derivation of formula (3) is not rigorous; in deriving this formula we first let $\Delta\rho_i$ approach zero, leaving $\Delta\theta_k$ constant, and only then made $\Delta\theta_k$ approach zero. This does not exactly correspond to the definition of a double integral, which we regard as the limit of an integral sum as the diameters of the subdomains approach zero (i.e., in the simultaneous approach to zero of $\Delta\theta_k$ and $\Delta\rho_i$). However, though the proof lacks rigour, the result is true [i.e., formula (3) is true]. This formula could be rigorously derived by the method used when considering the double integral in rectangular coordinates. We also note that this formula will be derived once again in Sec. 2.6 with different reasoning (as a particular case of the more general formula for transforming coordinates in a double integral).

Let it be required to compute the double integral of a function $f(x, y)$ over a domain $D$ given in rectangular coordinates:

$$\iint_D f(x, y)\,dx\,dy$$

If $D$ is regular in the polar coordinates $\theta$, $\rho$, then the computation of the given integral can be reduced to computing the iterated integral in polar coordinates.

Indeed, since

$$x = \rho\cos\theta, \quad y = \rho\sin\theta$$
$$f(x, y) = f[\rho\cos\theta, \rho\sin\theta] = F(\theta, \rho),$$

it follows that

$$\iint_D f(x, y)\,dx\,dy = \int_\alpha^\beta \left( \int_{\Phi_1(\theta)}^{\Phi_2(\theta)} f[\rho\cos\theta, \rho\sin\theta]\,\rho\,d\rho \right) d\theta \tag{4}$$

**Example 1.** Compute the volume $V$ of a solid bounded by the spherical surface

$$x^2 + y^2 + z^2 = 4a^2$$

and the cylinder

$$x^2 + y^2 - 2ay = 0$$

**Solution.** For the domain of integration here we can take the base of the cylinder $x^2 + y^2 - 2ay = 0$, that is, a circle with centre at $(0, a)$ and radius $a$. The equation of this circle may be written in the form $x^2 + (y-a)^2 = a^2$ (Fig. 65).

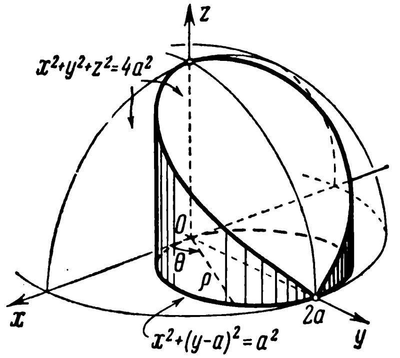


Fig. 65

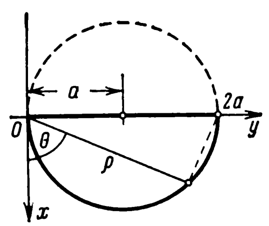


Fig. 66

We calculate $\frac{1}{4}$ of the required volume $V$, namely that part which is situated in the first octant. Then for the domain of integration we will have to take the semicircle whose boundaries are defined by the equations

$$x = \varphi_1(y) = 0, \quad x = \varphi_2(y) = \sqrt{2ay - y^2}$$
$$y = 0, \quad y = 2a$$

The integrand is

$$z = f(x, y) = \sqrt{4a^2 - x^2 - y^2}$$

Consequently

$$\frac{1}{4}V=\int_0^{2a}\left(\int_0^{\sqrt{2ay-y^2}}\sqrt{4a^2-x^2-y^2}\,dx\right)dy$$

Transform the integral obtained to the polar coordinates $\theta$, $\rho$:

$$x=\rho\cos\theta, \qquad y=\rho\sin\theta$$

Determine the limits of integration. To do so, write the equation of the given circle in polar coordinates; since

$$x^2+y^2=\rho^2$$
$$y=\rho\sin\theta$$

it follows that

$$\rho^2-2a\rho\sin\theta=0$$

or

$$\rho=2a\sin\theta$$

Hence, in polar coordinates (Fig. 66), the boundaries of the domain are defined by the equations

$$\rho=\Phi_1(\theta)=0, \quad \rho=\Phi_2(\theta)=2a\sin\theta, \quad \alpha=0, \quad \beta=\frac{\pi}{2}$$

and the integrand has the form

$$F(\theta,\ \rho)=\sqrt{4a^2-\rho^2}$$

Thus, we have

$$\frac{V}{4}=\int_0^{\frac{\pi}{2}}\left(\int_0^{2a\sin\theta}\sqrt{4a^2-\rho^2}\,\rho\,d\rho\right)d\theta=\int_0^{\frac{\pi}{2}}\left[-\frac{(4a^2-\rho^2)^{3/2}}{3}\right]_0^{2a\sin\theta}d\theta$$

$$=-\frac{1}{3}\int_0^{\frac{\pi}{2}}\left[(4a^2-4a^2\sin^2\theta)^{3/2}-(4a^2)^{3/2}\right]d\theta=\frac{8a^3}{3}\int_0^{\frac{\pi}{2}}(1-\cos^3\theta)\,d\theta$$

$$=\frac{4}{9}a^3(3\pi-4)$$

**Example 2.** Evaluate the Poisson integral

$$\int_{-\infty}^{+\infty}e^{-x^2}\,dx$$

**Solution.** First evaluate the integral $I_R=\iint\limits_D e^{-x^2-y^2}\,dx\,dy$, where the domain of integration $D$ is the circle $x^2+y^2=R^2$ (Fig. 67).

Passing to the polar coordinates $\theta$, $\rho$, we obtain

$$I_R=\int_0^{2\pi}\left(\int_0^R e^{-\rho^2}\rho\,d\rho\right)d\theta=-\frac{1}{2}\int_0^{2\pi}e^{-\rho^2}\Big|_0^R\,d\theta=\pi(1-e^{-R^2})$$

Now, if we increase the radius $R$ without bound (that is, if we expand without limit the domain of integration), we get the so-called *improper iterated integral:*

$$\int_0^{2\pi}\left(\int_a^{\infty} e^{-\rho^2}\rho\,d\rho\right)d\theta=\lim_{R\to\infty}\int_0^{2\pi}\left(\int_0^{R} e^{-\rho^2}\rho\,d\rho\right)d\theta=\lim_{R\to\infty}\pi\left(1-e^{-R^2}\right)=\pi$$

We shall show that the integral $\iint\limits_D e^{-x^2-y^2}\,dx\,dy$ approaches the limit $\pi$ if a domain $D'$ of arbitrary form expands in such manner that finally any point of the plane is in $D'$ and remains there (we shall conditionally indicate such an expansion of $D'$ by the relationship $D'\to\infty$).

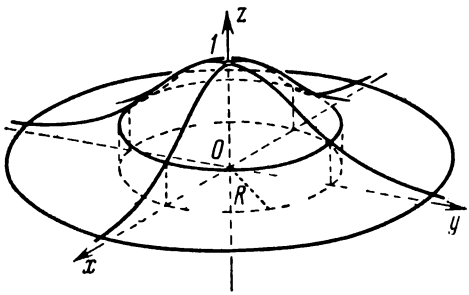


Fig. 67

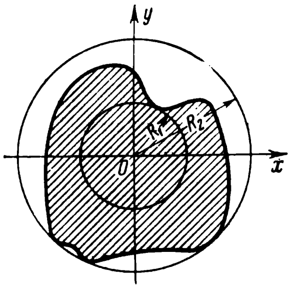


Fig. 68

Let $R_1$ and $R_2$ be the least and greatest distances of the boundary of $D'$ from the origin (Fig. 68).

Since the function $e^{-x^2-y^2}$ is everywhere greater than zero, the following inequalities hold:

$$I_{R_1}\leqslant\iint\limits_{D'} e^{-x^2-y^2}\,dx\,dy\leqslant I_{R_2}$$

or

$$\pi\left(1-e^{-R_1^2}\right)\leqslant\iint\limits_{D'} e^{-x^2-y^2}\,dx\,dy\leqslant\pi\left(1-e^{-R_2^2}\right)$$

Since as $D'\to\infty$ it is obvious that $R_1\to\infty$ and $R_2\to\infty$, it follows that the extreme parts of the inequality tend to the same limit $\pi$. Hence, the middle term also approaches this limit; that is,

$$\lim_{D'\to\infty}\iint\limits_{D'} e^{-x^2-y^2}\,dx\,dy=\pi \qquad (5)$$

As a particular instance, let $D'$ be a square with side $2a$ and centre at the origin; then

$$\iint\limits_{D'} e^{-x^2-y^2}\,dx\,dy=\int_{-a}^{a}\int_{-a}^{a} e^{-x^2-y^2}\,dx\,dy$$

$$=\int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\,dy=\int_{-a}^{a}\left(\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\right)dy$$

Now take the factor $e^{-y^2}$ outside the sign of the inner integral (this is permissible since $e^{-y^2}$ does not depend on the variable of integration $x$). Then

$$\iint_{D'} e^{-x^2-y^2}\,dx\,dy = \int_{-a}^{a} e^{-y^2}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)dy$$

Set $\int_{-a}^{a} e^{-x^2}\,dx = B_a$. This is a constant (dependent only on $a$); therefore,

$$\iint_{D'} e^{-x^2-y^2}\,dx\,dy = \int_{-a}^{a} e^{-y^2} B_a\,dy = B_a \int_{-a}^{a} e^{-y^2}\,dy$$

But the latter integral is likewise equal to $B_a$ $\left(\text{because } \int_{-a}^{a} e^{-x^2}\,dx = \int_{-a}^{a} e^{-y^2}\,dy\right)$; thus,

$$\iint_{D'} e^{-x^2-y^2}\,dx\,dy = B_a B_a = B_a^2$$

We pass to the limit in this equation, by making $a$ approach infinity (in the process, $D'$ expands without limit):

$$\lim_{D'\to\infty} \iint_{D'} e^{-x^2-y^2}\,dx\,dy = \lim_{a\to\infty} B_a^2 = \lim_{a\to\infty}\left[\int_{-a}^{a} e^{-x^2}\,dx\right]^2 = \left[\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right]^2$$

But, as has been proved [see (5)],

$$\lim_{D'\to\infty} \iint_{D'} e^{-x^2-y^2}\,dx\,dy = \pi$$

Hence,

$$\left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]^2 = \pi$$

or

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

This integral is frequently encountered in probability theory and in statistics. We remark that we would not be able to compute this integral directly (by means of an indefinite integral) because the antiderivative of $e^{-x^2}$ is not expressible in terms of elementary functions.

## 2.6 CHANGE OF VARIABLES IN A DOUBLE INTEGRAL (GENERAL CASE)

In the $xy$-plane let there be a domain $D$ bounded by a line $L$. Suppose that the coordinates $x$ and $y$ are functions of new variables $u$ and $v$:

$$x = \varphi(u, v), \quad y = \psi(u, v) \tag{1}$$

Let the functions $\varphi(u, v)$ and $\psi(u, v)$ be single-valued and continuous, and let them have continuous derivatives in some domain $D'$, which will be defined later on. Then by formulas (1) to each pair of values $u$ and $v$ there corresponds a unique pair of values $x$ and $y$. Further, suppose that the functions $\varphi$ and $\psi$ are such that if we give $x$ and $y$ definite values in $D$, then by formulas (1) we will find definite values of $u$ and $v$.

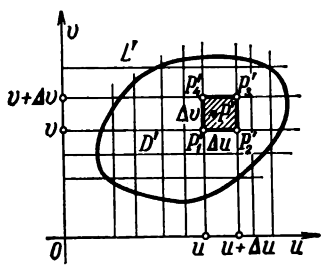


Fig. 69

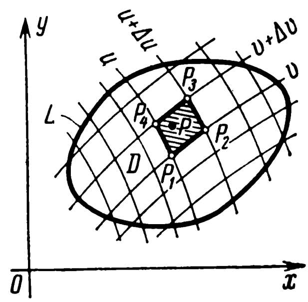


Fig. 70

Consider a rectangular coordinate system $Ouv$ (Fig. 69). From the foregoing it follows that with each point $P(x, y)$ in the $xy$-plane (Fig. 70) there is uniquely associated a point $P'(u, v)$ in the $uv$-plane with coordinates $u$, $v$, which are determined by formulas (1). The numbers $u$ and $v$ are called *curvilinear* coordinates of the point $P$.

If in the $xy$-plane a point describes a closed line $L$ bounding the domain $D$, then in the $uv$-plane a corresponding point will trace out a closed line $L'$ bounding a certain domain $D'$; and to each point of $D'$ there will correspond a point of $D$.

Thus, the formulas (1) establish a *one-to-one correspondence between the points of the domains D and D'*, or, the *mapping*, by formulas (1), *of D onto D' is said to be one-to-one.*

In the domain $D'$ let us consider a line $u = \text{const}$. By formulas (1) we find that in the $xy$-plane there will, generally speaking, be a certain curve corresponding to it. In exactly the same way, to each straight line $v = \text{const}$ of the $uv$-plane there will correspond some line in the $xy$-plane.

Let us divide $D'$ (using the straight lines $u=\text{const}$ and $v=\text{const}$) into rectangular subdomains (we shall disregard subdomains that overlap the boundary of the region $D'$). Using suitable curves, divide $D$ into certain curvilinear quadrangles (Fig. 70).

Consider, in the $uv$-plane, a rectangular subdomain $\Delta s'$ bounded by the straight lines $u=\text{const}$, $u+\Delta u=\text{const}$, $v=\text{const}$, $v+\Delta v=\text{const}$, and consider also the curvilinear subdomain $\Delta s$ corresponding to it in the $xy$-plane. We denote the areas of these subdomains by $\Delta s'$ and $\Delta s$, respectively. Then, obviously,

$$\Delta s' = \Delta u\, \Delta v$$

Generally speaking, the areas $\Delta s$ and $\Delta s'$ are different.

Suppose in $D$ we have a continuous function

$$z=f(x, y)$$

To each value of the function $z=f(x, y)$ in $D$ there corresponds the very same value of the function $z=F(u, v)$ in $D'$, where

$$F(u, v)=f[\varphi(u, v), \quad \psi(u, v)]$$

Consider the integral sums of the function $z$ over $D$. It is obvious that we have the following equation:

$$\sum f(x, y)\,\Delta s=\sum F(u, v)\,\Delta s \tag{2}$$

Let us compute $\Delta s$, which is the area of the curvilinear quadrangle $P_1P_2P_3P_4$ in the $xy$-plane (see Fig. 70).

We determine the coordinates of its vertices:

$$\left.\begin{array}{lll} P_1(x_1, y_1), & x_1=\varphi(u, v), & y_1=\psi(u, v) \\ P_2(x_2, y_2), & x_2=\varphi(u+\Delta u, v), & y_2=\psi(u+\Delta u, v) \\ P_3(x_3, y_3), & x_3=\varphi(u+\Delta u, v+\Delta v), & y_3=\psi(u+\Delta u, v+\Delta v) \\ P_4(x_4, y_4), & x_4=\varphi(u, v+\Delta v), & y_4=\psi(u, v+\Delta v) \end{array}\right\} \tag{3}$$

When computing the area of the curvilinear quadrangle $P_1P_2P_3P_4$ we shall consider the lines $P_1P_2$, $P_2P_3$, $P_3P_4$, $P_4P_1$ as parallel in pairs; we shall also replace the increments of the functions by corresponding differentials. We shall thus ignore infinitesimals of order higher than the infinitesimals $\Delta u$, $\Delta v$. Then formulas (3) will have the form

$$\left.\begin{array}{ll} x_1=\varphi(u, v), & y_1=\psi(u, v) \\ x_2=\varphi(u, v)+\frac{\partial\varphi}{\partial u}\Delta u, & y_2=\psi(u, v)+\frac{\partial\psi}{\partial u}\Delta u \\ x_3=\varphi(u, v)+\frac{\partial\varphi}{\partial u}\Delta u+\frac{\partial\varphi}{\partial v}\Delta v, & y_3=\psi(u, v)+\frac{\partial\psi}{\partial u}\Delta u+\frac{\partial\psi}{\partial v}\Delta v \\ x_4=\varphi(u, v)+\frac{\partial\varphi}{\partial v}\Delta v, & y_4=\psi(u, v)+\frac{\partial\psi}{\partial v}\Delta v \end{array}\right\} \tag{3'}$$

With these assumptions, the curvilinear quadrangle $P_1P_2P_3P_4$ may be regarded as a parallelogram. Its area $\Delta s$ is approximately equal to the doubled area of the triangle $P_1P_2P_3$ and is found by the following formula of analytic geometry:

$$\begin{aligned}\Delta s &\approx \left|(x_3-x_1)(y_3-y_2)-(x_3-x_2)(y_3-y_1)\right| \\ &= \left|\left(\frac{\partial\varphi}{\partial u}\Delta u+\frac{\partial\varphi}{\partial v}\Delta v\right)\frac{\partial\psi}{\partial v}\Delta v-\frac{\partial\varphi}{\partial v}\Delta v\left(\frac{\partial\psi}{\partial u}\Delta u+\frac{\partial\psi}{\partial v}\Delta v\right)\right| \\ &= \left|\frac{\partial\varphi}{\partial u}\frac{\partial\psi}{\partial v}\Delta u\,\Delta v-\frac{\partial\varphi}{\partial v}\frac{\partial\psi}{\partial u}\Delta u\Delta v\right| = \left|\frac{\partial\varphi}{\partial u}\frac{\partial\psi}{\partial v}-\frac{\partial\varphi}{\partial v}\frac{\partial\psi}{\partial u}\right|\Delta u\,\Delta v \\ &= \left\|\begin{matrix}\frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v}\\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v}\end{matrix}\right\|\Delta u\,\Delta v\end{aligned}$$

Here, the outer vertical lines indicate that the absolute value of the determinant is taken. We introduce the notation

$$\left|\begin{matrix}\frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v}\\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v}\end{matrix}\right| = I$$

Thus,

$$\Delta s \approx |I|\,\Delta s' \tag{4}$$

The determinant $I$ is called the *functional determinant* of the functions $\varphi(u, v)$ and $\psi(u, v)$. It is also called the *Jacobian* after the German mathematician Jacobi.

Equation (4) is only approximate, because in the process of computing the area of $\Delta s$ we neglected infinitesimals of higher order. However, the smaller the dimensions of the subdomains $\Delta s$ and $\Delta s'$, the more exact will this equation be. And it becomes absolutely exact in the limit, when the diameters of the subdomains $\Delta s$ and $\Delta s'$ approach zero:

$$|I| = \lim_{\text{diam}\,\Delta s'\to 0}\frac{\Delta s}{\Delta s'}$$

Let us now apply the equation obtained to an evaluation of the double integral. From (2) we can write

$$\sum f(x, y)\,\Delta s \approx \sum F(u, v)\,|I|\,\Delta s'$$

(the integral sum on the right is extended over the domain $D'$). Passing to the limit as diam $\Delta s' \to 0$, we get the exact equation

$$\iint_D f(x, y)\,dx\,dy = \iint_{D'} F(u, v)\,|I|\,du\,dv \tag{5}$$

This is the *formula for transformation of coordinates in a double integral*. It permits reducing the evaluation of a double integral over a domain $D$ to the computation of a double integral over a domain $D'$, which may simplify the problem. A rigorous proof of this formula was first given by the noted Russian mathematician M. V. Ostrogradsky.

**Note.** The transformation from rectangular coordinates to polar coordinates considered in the preceding section is a special case of change of variables in a double integral. Here, $u=\theta$, $v=\rho$:

$$x=\rho\cos\theta, \quad y=\rho\sin\theta$$

The curve $AB\,(\rho=\rho_1)$ in the $xy$-plane (Fig. 71) is transformed into the straight line $A'B'$ in the $\theta\rho$-plane (Fig. 72). The curve

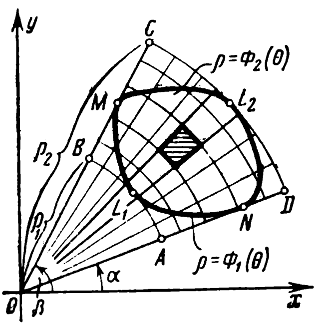


Fig. 71

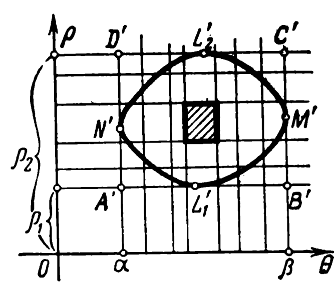


Fig. 72

$DC\,(\rho=\rho_2)$ in the $xy$-plane is transformed into the straight line $D'C'$ in the $\theta\rho$-plane.

The straight lines $AD$ and $BC$ in the $xy$-plane are transformed into the straight lines $A'D'$ and $B'C'$ in the $\theta\rho$-plane. The curves $L_1$ and $L_2$ are transformed into the curves $L_1'$ and $L_2'$.

Let us calculate the Jacobian of a transformation of the Cartesian coordinates $x$ and $y$ into the polar coordinates $\theta$ and $\rho$:

$$I=\begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\rho} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\rho} \end{vmatrix}=\begin{vmatrix} -\rho\sin\theta & \cos\theta \\ \rho\cos\theta & \sin\theta \end{vmatrix}=-\rho\sin^2\theta-\rho\cos^2\theta=-\rho$$

Hence, $|I|=\rho$ and therefore

$$\iint\limits_D f(x,\ y)\,dx\,dy=\int\limits_{\alpha}^{\beta}\left(\int\limits_{\Phi_1(\theta)}^{\Phi_2(\theta)} F(\theta,\ \rho)\,\rho\,d\rho\right)d\theta$$

This was the formula that we derived in the preceding section.

**Example.** Let it be required to compute the double integral

$$\iint_D (y-x)\,dx\,dy$$

over the region $D$ in the $xy$-plane bounded by the straight lines

$$y=x+1,\quad y=x-3,\quad y=-\frac{1}{3}x+\frac{7}{3},\quad y=-\frac{1}{3}x+5$$

It would be difficult to compute this double integral directly; however, a simple change of variables permits reducing this integral to one over a rectangle whose sides are parallel to the coordinate axes.

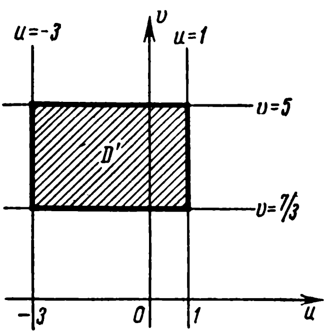


Fig. 73

Set

$$u=y-x,\quad v=y+\frac{1}{3}x \tag{6}$$

Then the straight lines $y=x+1$, $y=x-3$ will be transformed, respectively, into the straight lines $u=1$, $u=-3$ in the $uv$-plane; and the straight lines $y=-\frac{1}{3}x+\frac{7}{3}$, $y=-\frac{1}{3}x+5$ will be transformed into the straight lines $v=\frac{7}{3}$, $v=5$.

Consequently, the given domain $D$ is transformed into the rectangular domain $D'$ shown in Fig. 73. It remains to compute the Jacobian of the transformation. To do this, express $x$ and $y$ in terms of $u$ and $v$. Solving the system of equations (6), we obtain

$$x=-\frac{3}{4}u+\frac{3}{4}v,\quad y=\frac{1}{4}u+\frac{3}{4}v$$

Consequently,

$$I=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}-\frac{3}{4} & \frac{3}{4}\\ \frac{1}{4} & \frac{3}{4}\end{vmatrix}=-\frac{9}{16}-\frac{3}{16}=-\frac{3}{4}$$

and the absolute value of the Jacobian is $|I|=\frac{3}{4}$. Therefore,

$$\iint_D (y-x)\,dx\,dy=\iint_{D'}\left[\left(+\frac{1}{4}u+\frac{3}{4}v\right)-\left(-\frac{3}{4}u+\frac{3}{4}v\right)\right]\frac{3}{4}\,du\,dv$$

$$=\iint_{D'}\frac{3}{4}u\,du\,dv=\int_{\frac{7}{3}}^{5}\int_{-3}^{1}\frac{3}{4}u\,du\,dv=-8$$

## 2.7 COMPUTING THE AREA OF A SURFACE

Let it be required to compute the area of a surface bounded by a curve $\Gamma$ (Fig. 74); the surface is defined by the equation $z=f(x, y)$, where the function $f(x, y)$ is continuous and has continuous partial derivatives. Denote the projection of $\Gamma$ on the $xy$-plane by $L$. Denote by $D$ the domain on the $xy$-plane bounded by the curve $L$.

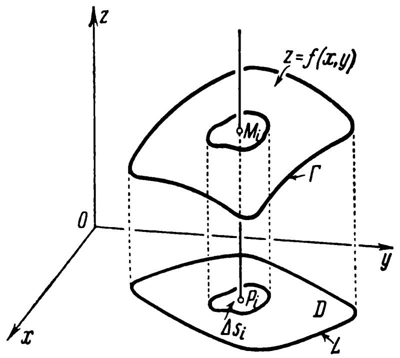


Fig. 74

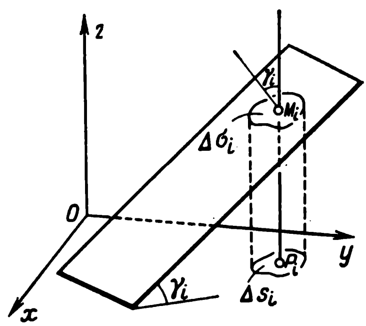


Fig. 75

In arbitrary fashion, divide $D$ into $n$ elementary subdomains $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. In each subdomain $\Delta s_i$ take a point $P_i(\xi_i, \eta_i)$. To the point $P_i$ there will correspond, on the surface, a point

$$M_i[\xi_i, \eta_i, f(\xi_i, \eta_i)]$$

Through $M_i$ draw a tangent plane to the surface. Its equation is of the form

$$z-z_i=f'_x(\xi_i, \eta_i)(x-\xi_i)+f'_y(\xi_i, \eta_i)(y-\eta_i) \tag{1}$$

(see Sec. 9.6, Vol. I). In this plane, pick out a subdomain $\Delta\sigma_i$ which is projected onto the $xy$-plane in the form of a subdomain $\Delta s_i$. Consider the sum of all the subdomains $\Delta\sigma_i$:

$$\sum_{i=1}^{n} \Delta\sigma_i$$

We shall call the limit $\sigma$ of this sum, when the greatest of the diameters of the subdomains $\Delta\sigma_i$ approaches zero, the *area of the surface*; that is, by definition we set

$$\sigma = \lim_{\operatorname{diam} \Delta\sigma_i \to 0} \sum_{i=1}^{n} \Delta\sigma_i \tag{2}$$

Now let us calculate the area of the surface. Denote by $\gamma_i$ the angle between the tangent plane and the $xy$-plane. Using a familiar formula of analytic geometry we can write (Fig. 75)

$$\Delta s_i = \Delta\sigma_i \cos\gamma_i$$

or

$$\Delta\sigma_i = \frac{\Delta s_i}{\cos\gamma_i} \tag{3}$$

The angle $\gamma_i$ is at the same time the angle between the $z$-axis and the perpendicular to the plane (1). Therefore, by equation (1) and the formula of analytic geometry we have

$$\cos\gamma_i = \frac{1}{\sqrt{1+f_x'^2(\xi_i, \eta_i)+f_y'^2(\xi_i, \eta_i)}}$$

Hence,

$$\Delta\sigma_i = \sqrt{1+f_x'^2(\xi_i, \eta_i)+f_y'^2(\xi_i, \eta_i)}\,\Delta s_i$$

Putting this expression into formula (2), we get

$$\sigma = \lim_{\operatorname{diam}\Delta s_i \to 0} \sum_{i=1}^{n} \sqrt{1+f_x'^2(\xi_i, \eta_i)+f_y'^2(\xi_i, \eta_i)}\,\Delta s_i$$

Since the limit of the integral sum on the right side of the last equation is, by definition, the double integral $\iint\limits_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$, we finally get

$$\sigma = \iint\limits_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy \tag{4}$$

This is the formula used to compute the area of the surface $z=f(x, y)$.

If the equation of the surface is given in the form

$$x=\mu(y, z) \text{ or in the form } y=\chi(x, z)$$

then the corresponding formulas for calculating the surface area are of the form

$$\sigma = \iint\limits_{D'} \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dy\,dz \tag{3'}$$

$$\sigma = \iint\limits_{D''} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dx\,dz \tag{3''}$$

where $D'$ and $D''$ are the domains in the $xy$-plane and the $xz$-plane in which the given surface is projected.

**Example 1.** Compute the surface area $\sigma$ of the sphere

$$x^2+y^2+z^2=R^2$$

**Solution.** Compute the surface area of the upper half of the sphere:

$$z=\sqrt{R^2-x^2-y^2}$$

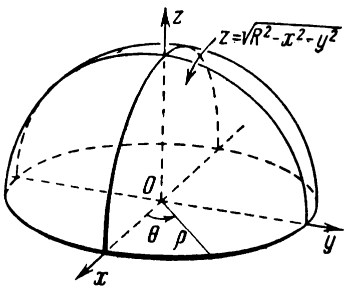


Fig. 76

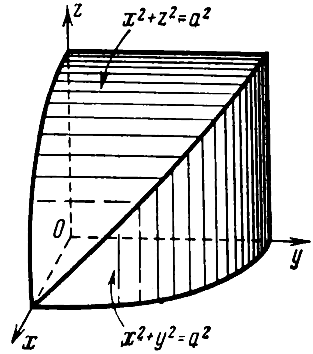


Fig. 77

Fig. 76). In this case

$$\frac{\partial z}{\partial x}=-\frac{x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y}=-\frac{y}{\sqrt{R^2-x^2-y^2}}$$

Hence,

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{\frac{R^2}{R^2-x^2-y^2}}=\frac{R}{\sqrt{R^2-x^2-y^2}}$$

The domain of integration is defined by the condition

$$x^2+y^2\leqslant R^2$$

Thus, by formula (4) we will have

$$\frac{1}{2}\sigma=\int_{-R}^{R}\left(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\frac{R}{\sqrt{R^2-x^2-y^2}}\,dy\right)dx$$

To compute the double integral obtained let us make the transformation to polar coordinates. In polar coordinates the boundary of the domain of integration is determined by the equation $\rho=R$. Hence,

$$\sigma=2\int_0^{2\pi}\left(\int_0^R\frac{R}{\sqrt{R^2-\rho^2}}\,\rho\,d\rho\right)d\theta=2R\int_0^{2\pi}\left[-\sqrt{R^2-\rho^2}\right]_0^R d\theta$$

$$=2R\int_0^{2\pi}R\,d\theta=4\pi R^2$$

**Example 2.** Find the area of that part of the surface of the cylinder

$$x^2+y^2=a^2$$

which is cut out by the cylinder

$$x^2+z^2=a^2$$

**Solution.** Fig. 77 shows 1/8th of the desired surface. The equation of the surface has the form $y=\sqrt{a^2-x^2}$; therefore,

$$\frac{\partial y}{\partial x}=-\frac{x}{\sqrt{a^2-x^2}}, \quad \frac{\partial y}{\partial z}=0$$

$$\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}=\sqrt{1+\frac{x^2}{a^2-x^2}}=\frac{a}{\sqrt{a^2-x^2}}$$

The domain of integration is a quarter of the circle, that is, it is determined by the conditions

$$x^2+z^2\leqslant a^2, \ x\geqslant 0, \ z\geqslant 0$$

Consequently,

$$\frac{1}{8}\sigma=\int_0^a\left(\int_0^{\sqrt{a^2-x^2}}\frac{a}{\sqrt{a^2-x^2}}\,dz\right)dx=a\int_0^a\frac{z}{\sqrt{a^2-x^2}}\Bigg|_0^{\sqrt{a^2-x^2}}dx=a\int_0^a dx=a^2$$

$$\sigma=8a^2$$

## 2.8 THE DENSITY DISTRIBUTION OF MATTER AND THE DOUBLE INTEGRAL

In a domain $D$, let a certain substance be distributed in such manner that there is a definite amount per unit area of $D$. We shall henceforward speak of the distribution of **mass**, although our reasoning will hold also for the case when speaking of the distribution of electric charge, quantity of heat, and so forth.

We consider an arbitrary subdomain $\Delta s$ of the domain $D$. Let the mass of substance associated with this given subdomain be $\Delta m$. Then the ratio $\frac{\Delta m}{\Delta s}$ is called the mean surface density of the substance in the subdomain $\Delta s$.

Now let the subdomain $\Delta s$ decrease and contract to the point $P(x, y)$. Consider the limit $\lim\limits_{\Delta s\to 0}\frac{\Delta m}{\Delta s}$. If this limit exists, then, generally speaking, it will depend on the position of the point $P$, that is, upon its coordinates $x$ and $y$, and will be some function $f(P)$ of the point $P$. We shall call this limit the *surface density* of the substance at the point $P$:

$$\lim_{\Delta s\to 0}\frac{\Delta m}{\Delta s}=f(P)=f(x, y) \qquad (1)$$

Thus, the surface density is a function $f(x, y)$ of the coordinates of the point of the domain.

Conversely, in a domain $D$, let the surface density of some substance be given as a continuous function $f(P)=f(x, y)$ and let it be required to determine the total quantity of substance $M$ contained in $D$. Divide $D$ into subdomains $\Delta s_i$ $(i=1, 2, \ldots, n)$ and in each subdomain take a point $P_i$; then $f(P_i)$ is the surface density at the point $P_i$.

To within higher-order infinitesimals, the product $f(P_i)\,\Delta s_i$ gives us the quantity of substance contained in the subdomain $\Delta s_i$, and the sum

$$\sum_{i=1}^{n} f(P_i)\,\Delta s_i$$

expresses approximately the total quantity of substance distributed in the domain $D$. But this is the integral sum of the function $f(P)$ in $D$. The exact value is obtained in the limit as $\Delta s_i \to 0$. Thus,*

$$M=\lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(P_i)\,\Delta s_i=\iint_D f(P)\,ds=\iint_D f(x, y)\,dx\,dy \qquad (2)$$

or the total quantity of substance in $D$ is equal to the double integral (over $D$) of the density $f(P)=f(x, y)$ of this substance.

**Example.** Determine the mass of a circular lamina of radius $R$ if the surface density $f(x, y)$ of the material at each point $P(x, y)$ is proportional to the distance of the point $(x, y)$ from the centre of the circle, that is, if

$$f(x, y)=k\sqrt{x^2+y^2}$$

**Solution.** By formula (2) we have

$$M=\iint_D k\sqrt{x^2+y^2}\,dx\,dy$$

where the domain of integration $D$ is the circle $x^2+y^2 \leqslant R^2$.

Passing to polar coordinates, we obtain

$$M=k\int_0^{2\pi}\left(\int_0^R \rho\rho\,d\rho\right)d\theta=k2\pi\frac{\rho^3}{3}\Bigg|_0^R=\frac{2}{3}k\pi R^3$$

## 2.9 THE MOMENT OF INERTIA OF THE AREA OF A PLANE FIGURE

*The moment of inertia $I$ of a material point $M$* of mass $m$ relative to some point $O$ is the product of the mass $m$ by the square of its distance $r$ from the point $O$:

$$I=mr^2$$

* The relationship $\Delta s_i \to 0$ is to be understood in the sense that the diameter of the subdomain $\Delta s_i$ approaches zero.

*The moment of inertia of a system of material points* $m_1$, $m_2$, $\ldots$, $m_n$ relative to $O$ is the sum of moments of inertia of the individual points of the system:

$$I = \sum_{i=1}^{n} m_i r_i^2$$

Let us determine the moment of inertia of a material plane figure $D$.

Let $D$ be located in an $xy$-coordinate plane. Let us determine the moment of inertia of this figure relative to the origin, assuming that the surface density is everywhere equal to unity.

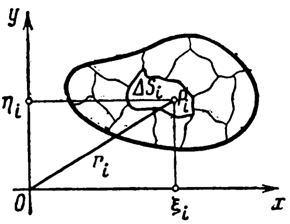

Fig. 78

Divide the domain $D$ into elementary subdomains $\Delta S_i$ ($i = 1, 2, \ldots, n$) (Fig. 78). In each subdomain take a point $P_i$ with coordinates $\xi_i$, $\eta_i$. Let us call the product of the mass of the subdomain $\Delta S_i$ by the square of the distance $r_i^2 = \xi_i^2 + \eta_i^2$ an elementary moment of inertia $\Delta I_i$ of the subdomain $\Delta S_i$:

$$\Delta I_i = (\xi_i^2 + \eta_i^2)\,\Delta S_i$$

and let us form the sum of such moments:

$$\sum_{i=1}^{n} (\xi_i^2 + \eta_i^2)\,\Delta S_i$$

This is the integral sum of the function $f(x, y) = x^2 + y^2$ over the domain $D$.

We define the *moment of inertia of the figure* $D$ as the limit of this sum when the diameter of each elementary subdomain $\Delta S_i$ approaches zero:

$$I_0 = \lim_{\operatorname{diam} \Delta S_i \to 0} \sum_{i=1}^{n} (\xi_i^2 + \eta_i^2)\,\Delta S_i$$

But the limit of this sum is the double integral $\iint\limits_D (x^2 + y^2)\,dx\,dy$.

Thus, the moment of inertia of the figure $D$ relative to the origin is

$$I_0 = \iint\limits_D (x^2 + y^2)\,dx\,dy \qquad (1)$$

where $D$ is a domain which coincides with the given plane figure.

The integrals

$$I_{xx} = \iint\limits_D y^2\,dx\,dy \qquad (2)$$

$$I_{yy} = \iint\limits_D x^2\,dx\,dy \qquad (3)$$

are called, respectively, the *moments of inertia of the figure D relative to the x-axis and y-axis.*

**Example 1.** Compute the moment of inertia of the area of a circle $D$ of radius $R$ relative to the centre $O$.

**Solution.** By formula (1) we have

$$I_0=\iint_D (x^2+y^2)\,dx\,dy$$

To evaluate this integral we change to the polar coordinates $\theta$, $\rho$. The equation of the circle in polar coordinates is $\rho=R$. Therefore

$$I_0=\int_0^{2\pi}\left(\int_0^R \rho^2\rho\,d\rho\right)d\theta=\frac{\pi R^4}{2}$$

**Note.** If the surface density $\gamma$ is not equal to unity, but is some function of $x$ and $y$, i.e., $\gamma=\gamma(x, y)$, then the mass of the subdomain $\Delta S_i$ will, to within infinitesimals of higher order, be equal to $\gamma(\xi_i, \eta_i)\,\Delta S_i$ and, for this reason, the moment of inertia of the plane figure relative to the origin will be

$$I_0=\iint_D \gamma(x, y)(x^2+y^2)\,dx\,dy \qquad (1')$$

**Example 2.** Compute the moment of inertia of a plane material figure $D$ bounded by the lines $y^2=1-x$; $x=0$, $y=0$ relative to the $y$-axis if the surface density at each point is equal to $y$ (Fig. 79).

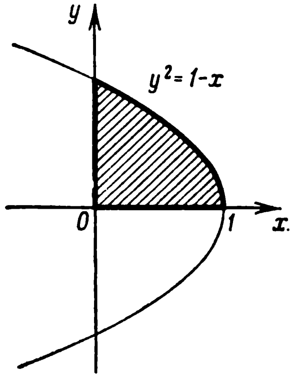


Fig. 79

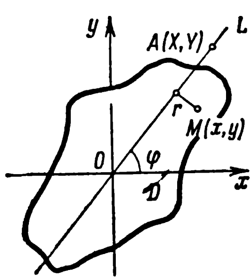


Fig. 80

**Solution.**

$$I_{yy}=\int_0^1\left(\int_0^{\sqrt{1-x}} yx^2\,dy\right)dx=\int_0^1 \frac{x^2y^2}{2}\Bigg|_0^{\sqrt{1-x}}dx=\frac{1}{2}\int_0^1 x^2(1-x)\,dx=\frac{1}{24}$$

**Ellipse of inertia.** Let us determine the moment of inertia of the area of a plane figure $D$ relative to some axis $OL$ that passes through the point $O$, which we shall take as the coordinate origin. Denote by $\varphi$ the angle formed by the straight line $OL$ with the positive $x$-axis (Fig. 80).

The normal equation of $OL$ is

$$x \sin\varphi - y\cos\varphi = 0$$

The distance $r$ of some point $M(x, y)$ from this line is

$$r = |x \sin\varphi - y\cos\varphi|$$

The moment of inertia $I$ of the area of $D$ relative to $OL$ is expressed, by definition, by the integral

$$I = \iint_D r^2\,dx\,dy = \iint_D (x\sin\varphi - y\cos\varphi)^2 dx\,dy$$

$$= \sin^2\varphi \iint_D x^2\,dx\,dy - 2\sin\varphi\cos\varphi \iint_D xy\,dx\,dy + \cos^2\varphi \iint_D y^2\,dx\,dy$$

Therefore

$$I = I_{yy}\sin^2\varphi - 2I_{xy}\sin\varphi\cos\varphi + I_{xx}\cos^2\varphi \tag{4}$$

here, $I_{yy} = \iint_D x^2\,dx\,dy$ is the moment of inertia of the figure relative to the $y$-axis, $I_{xx} = \iint_D y^2\,dx\,dy$ is the moment of inertia relative to the $x$-axis, and $I_{xy} = \iint_D xy\,dx\,dy$. Dividing all terms of the last equation by $I$, we get

$$1 = I_{xx}\left(\frac{\cos\varphi}{\sqrt{I}}\right)^2 - 2I_{xy}\left(\frac{\sin\varphi}{\sqrt{I}}\right)\left(\frac{\cos\varphi}{\sqrt{I}}\right) + I_{yy}\left(\frac{\sin\varphi}{\sqrt{I}}\right)^2 \tag{5}$$

On the straight line $OL$ take a point $A(X, Y)$ such that

$$OA = \frac{1}{\sqrt{I}}$$

To the various directions of the $OL$-axis, that is, to various values of the angle $\varphi$, there correspond different values $I$ and different points $A$. Let us find the locus of the points $A$. Obviously,

$$X = \frac{1}{\sqrt{I}}\cos\varphi, \quad Y = \frac{1}{\sqrt{I}}\sin\varphi$$

By virtue of (5), the quantities $X$ and $Y$ are connected by the relation

$$1 = I_{xx}X^2 - 2I_{xy}XY + I_{yy}Y^2 \tag{6}$$

Thus, the locus of points $A(X, Y)$ is a second-degree curve (6). We shall prove that this curve is an ellipse.

The following inequality established by the Russian mathematician Bunyakovsky * holds true:

$$\left(\iint_D xy\,dx\,dy\right)^2 < \left(\iint_D x^2\,dx\,dy\right)\left(\iint_D y^2\,dx\,dy\right)$$

or

$$I_{xx}I_{yy} - I_{xy}^2 > 0$$

Thus, the discriminant of the curve (6) is positive and, consequently, the curve is an ellipse (Fig. 81). This ellipse is called the **ellipse of inertia.** The notion of an ellipse of inertia is very important in mechanics.

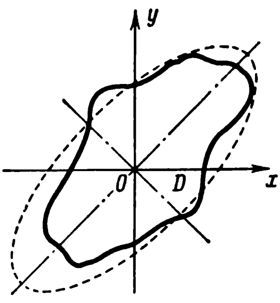

Fig. 81

We note that the lengths of the axes of the ellipse of inertia and its position in the plane depend on the shape of the given plane figure. Since the distance from the origin to some point $A$

* To prove Bunyakovsky's (also spelt Buniakowski) inequality, we consider the following obvious inequality:

$$\iint_D [f(x, y) - \lambda\varphi(x, y)]^2\,dx\,dy \geqslant 0$$

where $\lambda$ is a constant. The equality sign is possible only when $f(x, y) - \lambda\varphi(x, y) \equiv 0$; that is, if $f(x, y) = \lambda\varphi(x, y)$. If we assume that $\frac{f(x, y)}{\varphi(x, y)} \neq \text{const} = \lambda$, then the inequality sign will always hold. Thus, removing brackets under the integral sign, we obtain

$$\iint_D f^2(x, y)\,dx\,dy - 2\lambda\iint_D f(x, y)\,\varphi(x, y)\,dx\,dy + \lambda^2\iint_D \varphi^2(x, y)\,dx\,dy > 0$$

Consider the expression on the left as a function of $\lambda$. This is a second-degree polynomial that never vanishes; hence, its roots are complex, and this will occur when the discriminant formed of the coefficients of the quadratic polynomial is negative, that is,

$$\left(\iint_D f\varphi\,dx\,dy\right)^2 - \iint_D f^2\,dx\,dy\iint_D \varphi^2\,dx\,dy < 0$$

or

$$\left(\iint_D f\varphi\,dx\,dy\right)^2 < \iint_D f^2\,dx\,dy\iint_D \varphi^2\,dx\,dy$$

This is *Bunyakovsky's inequality*.

In our case, $f(x, y) = x$, $\varphi(x, y) = y$, $\frac{x}{y} \neq \text{const}$.

Bunyakovsky's inequality is widely used in various fields of mathematics. In many textbooks it is called Schwarz' inequality. Bunyakovsky published it (among other important inequalities) in 1859. Schwarz published his work 16 years later, in 1875.

of the ellipse is equal to $\frac{1}{\sqrt{I}}$, where $I$ is the moment of inertia of the figure relative to the $OA$-axis, it follows that, after constructing the ellipse, we can readily calculate the moment of inertia of the figure $D$ relative to some straight line passing through the coordinate origin. In particular, it is easy to see that the moment of inertia of the figure will be least relative to the major axis of the ellipse of inertia and greatest relative to the minor axis of this ellipse.

## 2.10 THE COORDINATES OF THE CENTRE OF GRAVITY OF THE AREA OF A PLANE FIGURE

In Sec. 12.8, Vol. I, it was stated that the coordinates of the centre of gravity of a system of material points $P_1, P_2, \ldots, P_n$ with masses $m_1, m_2, \ldots, m_n$ are defined by the formulas

$$x_C=\frac{\sum x_i m_i}{\sum m_i}, \quad y_C=\frac{\sum y_i m_i}{\sum m_i} \tag{1}$$

Let us now determine the coordinates of the centre of gravity of a plane figure $D$. Divide this figure into very small subdomains $\Delta S_i$. If the surface density is taken equal to unity, then the mass of a subdomain will be equal to its area. If it is approximately taken that the entire mass of subdomain $\Delta S_i$ is concentrated in some point of it, $P_i(\xi_i, \eta_i)$, the figure $D$ may be regarded as a **system of material points**. Then, by formulas (1), the coordinates of the centre of gravity of this figure will be **approximately** determined by the equations

$$x_C\approx\frac{\sum\limits_{i=1}^{n}\xi_i\Delta S_i}{\sum\limits_{i=1}^{n}\Delta S_i}; \quad y_C\approx\frac{\sum\limits_{i=1}^{n}\eta_i\Delta S_i}{\sum\limits_{i=1}^{n}\Delta S_i}$$

In the limit, as $\Delta S_i\to 0$, the integral sums in the numerators and denominators of the fractions will pass into double integrals, and we will obtain exact formulas for computing the coordinates of the centre of gravity of a plane figure:

$$x_C=\frac{\iint\limits_D x\,dx\,dy}{\iint\limits_D dx\,dy}, \quad y_C=\frac{\iint\limits_D y\,dx\,dy}{\iint\limits_D dx\,dy} \tag{2}$$

These formulas, which have been derived for a plane figure with surface density 1, obviously hold true also for a figure with any other density $\gamma$ constant at all points.

If, however, the surface density is variable,

$$\gamma = \gamma(x, y)$$

then the corresponding formulas will have the form

$$x_C = \frac{\iint\limits_D \gamma(x, y)\, x\, dx\, dy}{\iint\limits_D \gamma(x, y)\, dx\, dy}, \quad y_C = \frac{\iint\limits_D \gamma(x, y)\, y\, dx\, dy}{\iint\limits_D \gamma(x, y)\, dx\, dy}$$

The expressions $M_y = \iint\limits_D \gamma(x, y)\, x\, dx\, dy$ and $M_x = \iint\limits_D \gamma(x, y)\, y\, dx\, dy$ are called *static moments* of the plane figure $D$ relative to the $y$-axis and $x$-axis.

The integral $\iint \gamma(x, y)\, dx\, dy$ expresses the quantity of **mass** of the figure in question.

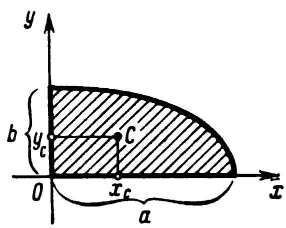

Fig. 82

**Example.** Determine the coordinates of the centre of gravity of a quarter of the ellipse (Fig. 82)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

assuming that the surface density at all points is equal to 1.

**Solution.** By formulas (2) we have

$$x_C = \frac{\int\limits_0^a \left( \int\limits_0^{\frac{b}{a}\sqrt{a^2-x^2}} x\, dy \right) dx}{\int\limits_0^a \left( \int\limits_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy \right) dx} = \frac{\frac{b}{a}\int\limits_0^a \sqrt{a^2-x^2}\, x\, dx}{\frac{1}{4}\pi ab} = \frac{-\frac{b}{a}\cdot\frac{1}{3}(a^2-x^2)^{3/2}\Big|_0^a}{\frac{1}{4}\pi ab} = \frac{4a}{3\pi}$$

$$y_C = \frac{\int\limits_0^a \left( \int\limits_0^{\frac{b}{a}\sqrt{a^2-x^2}} y\, dy \right) dx}{\frac{1}{4}\pi ab} = \frac{4b}{3\pi}$$

## 2.11 TRIPLE INTEGRALS

Let there be given, in space, a certain domain $V$ bounded by a closed surface $S$. Let some continuous function $f(x, y, z)$, where $x, y, z$ are the rectangular coordinates of a point of the domain, be given in $V$ and on its boundary. For clarity, if $f(x, y, z) \geqslant 0$, we can regard this function as the density distribution of some substance in the domain $V$.

Divide $V$, in arbitrary fashion, into subdomains $\Delta v_i$; the symbol $\Delta v_i$ will denote not only the domain itself, but its volume as well. Within the limits of each subdomain $\Delta v_i$, choose an arbitrary point $P_i$ and denote by $f(P_i)$ the value of the function $f$ at this point. Form a sum of the type

$$\sum f(P_i)\,\Delta v_i \tag{1}$$

and increase without bound the number of subdomains $\Delta v_i$ so that the largest diameter of $\Delta v_i$ should approach zero.* If the function $f(x, y, z)$ is continuous, sums of type (1) will have a limit. This limit is to be understood in the same sense as for the definition of the double integral.** It is not dependent either on the manner of partitioning the domain $V$ or on the choice of points $P_i$; it is designated by the symbol $\iiint\limits_V f(P)\,dv$ and is called a *triple integral*. Thus, by definition,

$$\lim_{\operatorname{diam} \Delta v_i \to 0} \sum f(P_i)\,\Delta v_i = \iiint\limits_V f(P)\,dv$$

or

$$\iiint\limits_V f(P)\,dv = \iiint\limits_V f(x.\ y,\ z)\,dx\,dy\,dz \tag{2}$$

If $f(x, y, z)$ is considered the volume density of distribution of a substance over the domain $V$, then the integral (2) yields the mass of the entire substance contained in $V$.

## 2.12 EVALUATING A TRIPLE INTEGRAL

Suppose that a spatial (three-dimensional) domain $V$ bounded by a closed surface $S$ possesses the following properties:

(1) every straight line parallel to the $z$-axis and drawn through an interior (that is, not lying on the boundary $S$) point of the domain $V$ cuts the surface $S$ at two points;

(2) the entire domain $V$ is projected on the $xy$-plane into a regular (two-dimensional) domain $D$;

(3) any part of the domain $V$ cut off by a plane parallel to any one of the coordinate planes ($Oxy$, $Oxz$, $Qyz$) likewise possesses Properties 1 and 2.

---

* The diameter of a subdomain $\Delta v_i$ is the maximum distance between points lying on the boundary of the subdomain.

** This theorem of the existence of a limit of integral sums (that is, of the existence of a triple integral) for any function continuous in a closed domain $V$ (including the boundary) is accepted without proof.

We shall call the domain $V$ that possesses the indicated properties a *regular* three-dimensional domain.

To illustrate, an ellipsoid, a rectangular parallelepiped, a tetrahedron, and so on are examples of regular three-dimensional domains. An instance of an irregular three-dimensional domain is given in Fig. 83. In this section we will consider only regular domains.

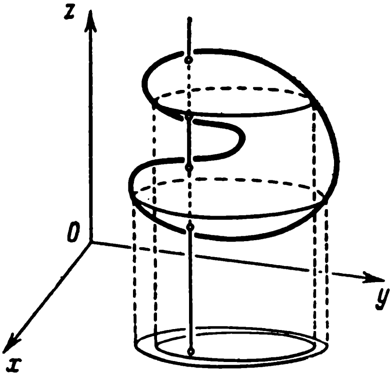


Fig. 83

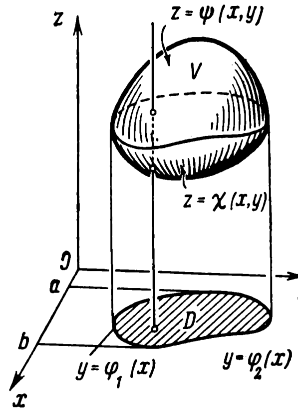


Fig. 84

Let the surface bounding $V$ below have the equation $z=\chi(x, y)$, and the surface bounding this domain above, the equation $z=\psi(x, y)$ (Fig. 84).

We introduce the concept of a **threefold iterated** integral $I_V$, over the domain $V$, of a function of three variables $f(x, y, z)$ defined and continuous in $V$. Suppose that the domain $D$ is the projection of the domain $V$ onto the $xy$-plane bounded by the curves

$$y=\varphi_1(x), \quad y=\varphi_2(x), \quad x=a, \quad x=b$$

Then *a threefold iterated integral* of the function $f(x, y, z)$ over $V$ is defined as follows:

$$I_V=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\left\{\int_{\chi(x, y)}^{\psi(x, y)} f(x, y, z)\,dz\right\}dy\right]dx \qquad (1)$$

We note that as a result of integration with respect to $z$ and substitution of limits in the braces (inner brackets) we get a function of $x$ and $y$. We then compute the double integral of this function over the domain $D$ as has already been done.

The following is an example of the evaluation of a threefold iterated integral.

**Example 1.** Compute the threefold iterated integral of the function $f(x, y, z) = xyz$ over the domain $V$ bounded by the planes

$$x=0, \quad y=0, \quad z=0, \quad x+y+z=1$$

**Solution.** This domain is regular, it is bounded above and below by the planes $z=0$ and $z=1-x-y$ and is projected on the $xy$-plane into a regular plane domain $D$, which is a triangle bounded by the straight lines $x=0$, $y=0$, $y=1-x$ (Fig. 85). Therefore, the threefold iterated integral $I_V$ is computed as follows:

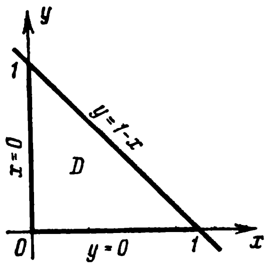


Fig. 85

$$I_V = \iint_D \left[ \int_0^{1-x-y} xyz\,dz \right] d\sigma$$

Setting up the limits in the twofold iterated integral over the domain $D$, we obtain

$$I_V = \int_0^1 \left\{ \int_0^{1-x} \left[ \int_0^{1-x-y} xyz\,dz \right] dy \right\} dx = \int_0^1 \left\{ \int_0^{1-x} \frac{xyz^2}{2} \Bigg|_{z=0}^{z=1-x-y} dy \right\} dx$$

$$= \int_0^1 \left\{ \int_0^{1-x} \frac{1}{2} xy(1-x-y)^2\,dy \right\} dx = \int_0^1 \frac{x}{24}(1-x)^4\,dx = \frac{1}{720}$$

Let us now consider some of the properties of a threefold iterated integral.

**Property 1.** *If a domain $V$ is divided into two domains $V_1$ and $V_2$ by a plane parallel to one of the coordinate planes, then the threefold iterated integral over $V$ is equal to the sum of the threefold iterated integrals over the domains $V_1$ and $V_2$.*

The proof of this property is exactly the same as that for twofold iterated integrals. We shall not repeat it.

**Corollary.** *For any kind of partition of the domain $V$ into a finite number of subdomains $V_1, \ldots, V_n$ by planes parallel to the coordinate planes, we have the equality*

$$I_V = I_{V_1} + I_{V_2} + \ldots + I_{V_n}$$

**Property 2 (Theorem on the evaluation of a threefold iterated integral).** *If $m$ and $M$ are, respectively, the smallest and largest values of the function $f(x, y, z)$ in the domain $V$, we have the inequality*

$$mV \leqslant I_V \leqslant MV$$

*where $V$ is the volume of the given domain and $I_V$ is a threefold iterated integral of the function $f(x, y, z)$ over $V$.*

**Proof.** Let us first evaluate the inside integral in the iterated integral $I_V=\iint\limits_D\left[\int\limits_{\chi(x,\,y)}^{\psi(x,\,y)} f(x,\,y,\,z)\,dz\right]d\sigma$:

$$\int\limits_{\chi(x,\,y)}^{\psi(x,\,y)} f(x,\,y,\,z)\,dz \leqslant \int\limits_{\chi(x,\,y)}^{\psi(x,\,y)} M\,dz = M\int\limits_{\chi(x,\,y)}^{\psi(x,\,y)} dz = Mz\Big|_{\chi(x,\,y)}^{\psi(x,\,y)} = M\left[\psi(x,\,y)-\chi(x,\,y)\right]$$

Thus, the inside integral does not exceed the expression $M[\psi(x,\,y)-\chi(x,\,y)]$. Therefore, by virtue of the theorem of Sec. 2.1 on double integrals, we get (denoting by $D$ the projection of the domain $V$ on the $xy$-plane)

$$I_V=\iint\limits_D\left[\int\limits_{\chi(x,\,y)}^{\psi(x,\,y)} f(x,\,y,\,z)\,dz\right]d\sigma \leqslant \iint\limits_D M\left[\psi(x,\,y)-\chi(x,\,y)\right]d\sigma = M\iint\limits_D\left[\psi(x,\,y)-\chi(x,\,y)\right]d\sigma$$

But the latter iterated integral is equal to the double integral of the function $\psi(x,\,y)-\chi(x,\,y)$ and, consequently, is equal to the volume of the domain which lies between the surfaces $z=\chi(x,\,y)$ and $z=\psi(x,\,y)$, that is, to the volume of the domain $V$. Therefore,

$$I_V \leqslant MV$$

It is similarly proved that $I_V \geqslant mV$. Property 2 is thus proved.

**Property 3 (Mean-value theorem).** *The threefold iterated integral $I_V$ of a continuous function $f(x,\,y,\,z)$ over a domain $V$ is equal to the product of its volume $V$ by the value of the function at some point $P$ of $V$; that is,*

$$I_V=\int\limits_a^b\left\{\int\limits_{\varphi_1(x)}^{\varphi_2(x)}\left[\int\limits_{\chi(x,\,y)}^{\psi(x,\,y)} f(x,\,y,\,z)\,dz\right]dy\right\}dx=f(P)\,V \tag{2}$$

The proof of this property is carried out in the same way as that for a twofold iterated integral [see Sec. 2.2, Property 3, formula (4)].

We can now prove the theorem for evaluating a triple integral.

**Theorem.** *The triple integral of a function $f(x,\,y,\,z)$ over a regular domain $V$ is equal to a threefold iterated integral over the same domain; that is,*

$$\iiint\limits_V f(x,\,y,\,z)\,dv=\int\limits_a^b\left\{\int\limits_{\varphi_1(x)}^{\varphi_2(x)}\left[\int\limits_{\chi(x,\,y)}^{\psi(x,\,y)} f(x,\,y,\,z)\,dz\right]dy\right\}dx$$

**Proof.** Divide the domain $V$ by planes parallel to the coordinate planes into $n$ regular subdomains:

$$\Delta v_1, \Delta v_2, \ldots, \Delta v_n$$

As done above, denote by $I_V$ the threefold iterated integral of the function $f(x, y, z)$ over the domain $V$, and by $I_{\Delta v_i}$ the threefold iterated integral of this function over the subdomain $\Delta v_i$. Then by the corollary of Property 1 we can write the equation

$$I_V = I_{\Delta v_1} + I_{\Delta v_2} + \ldots + I_{\Delta v_n} \tag{3}$$

We transform each of the terms on the right by formula (2):

$$I_V = f(P_1)\Delta v_1 + f(P_2)\Delta v_2 + \ldots + f(P_n)\Delta v_n \tag{4}$$

where $P_i$ is some point of the subdomain $\Delta v_i$.

On the right side of this equation is an integral sum. It is assumed that the function $f(x, y, z)$ is continuous in $V$; and for this reason the limit of this sum, as the largest diameter of $\Delta v_i$ approaches zero, exists and is equal to the triple integral of the function $f(x, y, z)$ over $V$. Thus, passing to the limit in (4), as diam $\Delta v_i \to 0$, we get

$$I_V = \iiint_V f(x, y, z)\,dv$$

or, finally, interchanging the expressions on the right and left,

$$\iiint_V f(x, y, z)\,dv = \int_a^b \left\{ \int_{\varphi_1(x)}^{\varphi_2(x)} \left[ \int_{\chi(x, y)}^{\psi(x, y)} f(x, y, z)\,dz \right] dy \right\} dx$$

Thus, the theorem is proved.

Here, $z = \chi(x, y)$ and $z = \psi(x, y)$ are the equations of the surfaces bounding the regular domain $V$ below and above. The lines $y = \varphi_1(x)$, $y = \varphi_2(x)$, $x = a$, $x = b$ bound the domain $D$, which is the projection of $V$ onto the $xy$-plane.

**Note.** As in the case of the double integral, we can form a threefold iterated integral with a different order of integration with respect to the variables and with other limits, if, of course, the shape of the domain $V$ permits this.

**Computing the volume of a solid by means of a threefold iterated integral.** If the integrand $f(x, y, z) = 1$, then the triple integral over the domain $V$ expresses the **volume of** $V$:

$$V = \iiint_V dx\,dy\,dz \tag{5}$$

**Example 2.** Compute the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

**Solution.** The ellipsoid (Fig. 86) is bounded below by the surface $z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$, and above by the surface $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$. The projection of this ellipsoid on the $xy$-plane (domain $D$) is an ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Hence, reducing the computation of volume to that of a threefold iterated integral, we obtain

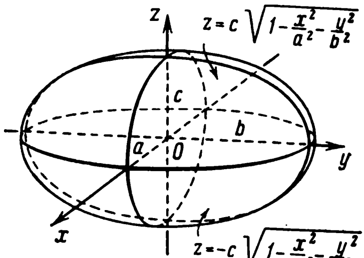


Fig. 86

$$V=\int_{-a}^{a}\left[\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}\left(\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dz\right)dy\right]dx$$

$$=2c\int_{-a}^{a}\left[\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\right]dx$$

When computing the inside integral, $x$ is held constant. Make the substitution:

$$y=b\sqrt{1-\frac{x^2}{a^2}}\sin t,\qquad dy=b\sqrt{1-\frac{x^2}{a^2}}\cos t\,dt$$

The variable $y$ varies from $-b\sqrt{1-\frac{x^2}{a^2}}$ to $b\sqrt{1-\frac{x^2}{a^2}}$; therefore $t$ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Putting new limits in the integral, we get

$$V=2c\int_{-a}^{a}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\left(1-\frac{x^2}{a^2}\right)-\left(1-\frac{x^2}{a^2}\right)\sin^2 t}\;b\sqrt{1-\frac{x^2}{a^2}}\cos t\,dt\right]dx$$

$$=2cb\int_{-a}^{a}\left[\left(1-\frac{x^2}{a^2}\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt\right]dx=\frac{cb\pi}{a^2}\int_{-a}^{a}(a^2-x^2)\,dx=\frac{4\pi abc}{3}$$

Hence,

$$V = \frac{4}{3}\pi abc$$

If $a = b = c$, we get the volume of the sphere:

$$V = \frac{4}{3}\pi a^3$$

## 2.13 CHANGE OF VARIABLES IN A TRIPLE INTEGRAL

**1. Triple integral in cylindrical coordinates.** In the case of cylindrical coordinates, the position of a point $P$ in space is determined by three numbers $\theta$, $\rho$, $z$, where $\theta$ and $\rho$ are polar coordinates of the projection of the point $P$ on the $xy$-plane and $z$ is the $z$-coordinate of $P$, that is, the distance of the point to the $xy$-

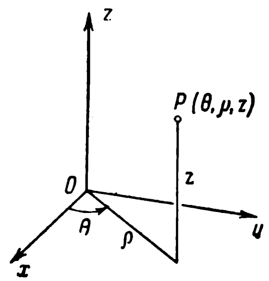


Fig. 87

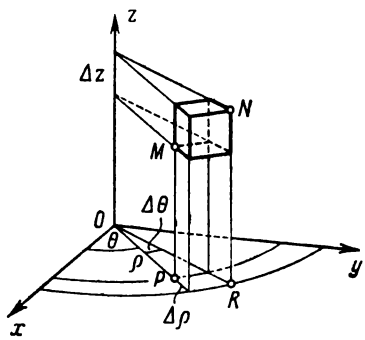


Fig. 88

plane—with the plus sign if the point lies above the $xy$-plane, and with the minus sign if below the $xy$-plane (Fig. 87).

In this case, we divide the given three-dimensional domain $V$ into elementary volumes by the coordinate surfaces $\theta = \theta_i$, $\rho = \rho_j$, $z = z_k$ (half-planes adjoining the $z$-axis, circular cylinders whose axis coincides with the $z$-axis, planes perpendicular to the $z$-axis). The curvilinear "prism" shown in Fig. 88 is a volume element. The base area of this prism is equal, to within infinitesimals of higher order, to $\rho\Delta\theta\,\Delta\rho$, the altitude is $\Delta z$ (to simplify notation we drop the indices $i$, $j$, $k$). Thus, $\Delta v = \rho\Delta\theta\,\Delta\rho\,\Delta z$. Hence, the triple integral of the function $F(\theta, \rho, z)$ over the domain $V$ has the form

$$I = \iiint_V F(\theta, \rho, z)\,\rho\, d\theta\, d\rho\, dz \qquad (1)$$

The limits of integration are determined by the shape of the domain $V$.

If a triple integral of the function $f(x, y, z)$ is given in rectangular coordinates, it can readily be changed to a triple integral in cylindrical coordinates. Indeed, noting that

$$x=\rho\cos\theta,\quad y=\rho\sin\theta,\quad z=z$$

we have

$$\iiint\limits_V f(x, y, z)\,dx\,dy\,dz=\iiint\limits_V F(\theta, \rho, z)\,\rho\,d\theta\,d\rho\,dz$$

where

$$f(\rho\cos\theta, \rho\sin\theta, z)=F(\theta, \rho, z)$$

**Example.** Determine the mass $M$ of a hemisphere of radius $R$ with centre at the origin, if the density $F$ of its substance at each point $(x, y, z)$ is proportional to the distance of this point from the base, that is, $F=kz$.

**Solution.** The equation of the upper part of the hemisphere

$$z=\sqrt{R^2-x^2-y^2}$$

in cylindrical coordinates has the form

$$z=\sqrt{R^2-\rho^2}$$

Hence,

$$M=\iiint\limits_V kz\rho\,d\theta\,d\rho\,dz=\int\limits_0^{2\pi}\left[\int\limits_0^R\left(\int\limits_0^{\sqrt{R^2-\rho^2}} kz\,dz\right)\rho\,d\rho\right]d\theta$$

$$=\int\limits_0^{2\pi}\left[\int\limits_0^R \frac{kz^2}{2}\Bigg|_0^{\sqrt{R^2-\rho^2}}\rho\,d\rho\right]d\theta=\int\limits_0^{2\pi}\left[\int\limits_0^R \frac{k}{2}(R^2-\rho^2)\,\rho\,d\rho\right]d\theta$$

$$=\frac{k}{2}\int\limits_0^{2\pi}\left[\frac{R^4}{2}-\frac{R^4}{4}\right]d\theta=\frac{k}{2}\,\frac{R^4}{4}\,2\pi=\frac{k\pi R^4}{4}$$

**2. Triple integral in spherical coordinates.** In spherical coordinates, the position of a point $P$ in space is determined by three numbers, $\theta$, $r$, $\varphi$, where $r$ is the distance of the point from the origin, the so-called radius vector of the point, $\varphi$ is the angle between the radius vector and the $z$-axis, $\theta$ is the angle between the projection of the radius vector on the $xy$-plane and the $x$-axis reckoned from this axis in a positive sense (counterclockwise) (Fig. 89). For any point of space we have

$$0\leqslant r<\infty,\quad 0\leqslant\varphi\leqslant\pi;\quad 0\leqslant\theta\leqslant 2\pi$$

Divide the domain $V$ into volume elements $\Delta v$ by the coordinate surfaces $r=\text{const}$ (spheres), $\varphi=\text{const}$ (conic surfaces with vertices at origin), $\theta=\text{const}$ (half-planes passing through the $z$-axis). To within infinitesimals of higher order, the volume element $\Delta v$ may be considered a parallelepiped with edges of length $\Delta r$, $r\Delta\varphi$,

$r \sin\varphi\, \Delta\theta$. Then the volume element is equal (see Fig. 90) to

$$\Delta v = r^2 \sin\varphi\, \Delta r\, \Delta\theta\, \Delta\varphi$$

The triple integral of a function $F(\theta, r, \varphi)$ over a domain $V$ has the form

$$I = \iiint_V F(\theta, r, \varphi)\, r^2 \sin\varphi\, dr\, d\theta\, d\varphi \qquad (1')$$

The limits of integration are determined by the shape of the domain $V$. From Fig. 89 it is easy to establish the expressions of

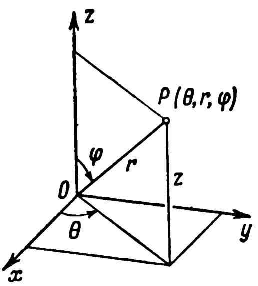


Fig. 89

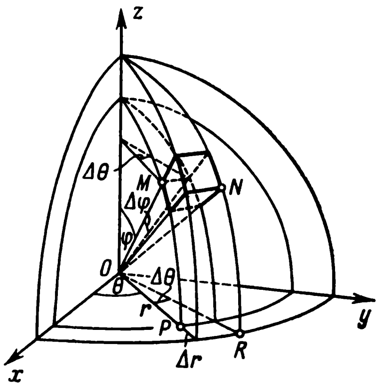


Fig. 90

Cartesian coordinates in terms of spherical coordinates:

$$x = r \sin\varphi \cos\theta$$
$$y = r \sin\varphi \sin\theta$$
$$z = r \cos\varphi$$

For this reason, the formula for transforming a triple integral from Cartesian coordinates to spherical coordinates has the form

$$\iiint_V f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint_V f[r \sin\varphi \cos\theta,\ r \sin\varphi \sin\theta,\ r\cos\varphi]\, r^2 \sin\varphi\, dr\, d\theta\, d\varphi$$

**3. General change of variables in a triple integral.** Transformations from Cartesian coordinates to cylindrical and spherical coordinates in a triple integral represent special cases of the general transformation of coordinates in space.

Let the functions

$$x = \varphi(u, t, w)$$
$$y = \psi(u, t, w)$$
$$z = \chi(u, t, w)$$

map, in one-to-one manner, a domain $V$ in Cartesian coordinates $x$, $y$, $z$ onto a domain $V'$ in curvilinear coordinates $u$, $t$, $w$. Let the volume element $\Delta v$ of $V$ be carried over to the volume element $\Delta v'$ of $V'$ and let

$$\lim_{\Delta v' \to 0} \frac{\Delta v}{\Delta v'} = |I|$$

Then

$$\iiint_V f(x, y, z)\,dx\,dy\,dz$$

$$= \iiint_{V'} f[\varphi(u, t, w), \psi(u, t, w), \chi(u, t, w)]\,|I|\,du\,dt\,dw$$

As in the case of the double integral, $I$ is called the *Jacobian*; and as in the case of double integrals, it may be proved that the Jacobian is numerically equal to a determinant of order three:

$$I = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Thus, in the case of cylindrical coordinates we have

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = z \quad (\rho = u, \ \theta = t, \ z = w)$$

$$I = \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

In the case of spherical coordinates we have

$$x = r\sin\varphi\cos\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\varphi \quad (r = u, \ \varphi = t, \ \theta = w)$$

$$I = \begin{vmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{vmatrix} = r^2\sin\varphi$$

## 2.14 THE MOMENT OF INERTIA AND THE COORDINATES OF THE CENTRE OF GRAVITY OF A SOLID

**1. The moment of inertia of a solid.** The moments of inertia of a **point** $M(x, y, z)$ of mass $m$ relative to the coordinate axes $Ox$, $Oy$ and $Oz$ (Fig. 91) are expressed, respectively, by the formulas

$$I_{xx} = (y^2 + z^2)\,m$$

$$I_{yy} = (x^2 + z^2)\,m, \quad I_{zz} = (x^2 + y^2)\,m$$

The moments of inertia of a **solid** are expressed by the corresponding integrals. For instance, the moment of inertia of a solid relative to the $z$-axis is expressed by the integral $I_{zz} = \iiint_V (x^2+y^2)\gamma(x, y, z)\,dx\,dy\,dz$, where $\gamma(x, y, z)$ is the density of the substance.

**Example 1.** Compute the moment of inertia of a right circular cylinder of altitude $2h$ and radius $R$ relative to the diameter of its median section, considering the density constant and equal to $\gamma_0$.

**Solution.** Choose a coordinate system as follows: direct the $z$-axis along the axis of the cylinder, and put the origin of coordinates at its centre of symmetry (Fig. 92).

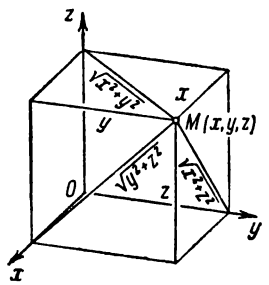


Fig. 91

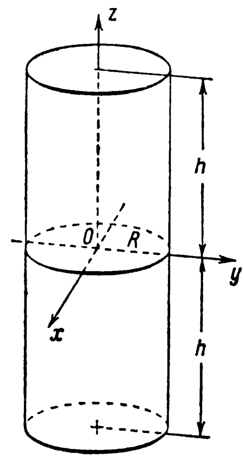


Fig. 92

Then the problem reduces to computing the moment of inertia of the cylinder relative to the $x$-axis:

$$I_{xx} = \iiint_V (y^2+z^2)\,\gamma_0\,dx\,dy\,dz$$

Changing to cylindrical coordinates, we obtain

$$I_{xx} = \gamma_0 \int_0^{2\pi} \left\{ \int_0^R \left[ \int_{-h}^{h} (z^2+\rho^2\sin^2\theta)\,dz \right] \rho\,d\rho \right\} d\theta$$

$$= \gamma_0 \int_0^{2\pi} \left\{ \int_0^R \left[ \frac{2h^3}{3} + 2h\rho^2\sin^2\theta \right] \rho\,d\rho \right\} d\theta = \gamma_0 \int_0^{2\pi} \left\{ \frac{2h^3}{3}\frac{R^2}{2} + \frac{2hR^4}{4}\sin^2\theta \right\} d\theta$$

$$= \gamma_0 \left[ \frac{2h^3R^2}{6}2\pi + \frac{2hR^4}{4}\pi \right] = \gamma_0\pi hR^2 \left[ \frac{2}{3}h^2 + \frac{R^2}{2} \right]$$

**2. The coordinates of the centre of gravity of a solid.** Like what we had in Sec. 12.8, Vol. I, for plane figures, the coordinates of the centre of gravity of a solid are expressed by the formulas

$$x_C = \frac{\iiint\limits_V x\gamma(x, y, z)\,dx\,dy\,dz}{\iiint\limits_V \gamma(x, y, z)\,dx\,dy\,dz}, \quad y_C = \frac{\iiint\limits_V y\gamma(x, y, z)\,dx\,dy\,dz}{\iiint\limits_V \gamma(x, y, z)\,dx\,dy\,dz}$$

$$z_C = \frac{\iiint\limits_V z\gamma(x, y, z)\,dx\,dy\,dz}{\iiint\limits_V \gamma(x, y, z)\,dx\,dy\,dz}$$

where $\gamma(x, y, z)$ is the density.

**Example 2.** Determine the coordinates of the centre of gravity of the upper half of a sphere of radius $R$ with centre at the origin, assuming the density $\gamma_0$ constant.

**Solution.** The hemisphere is bounded by the surfaces

$$z = \sqrt{R^2 - x^2 - y^2}, \quad z = 0$$

The $z$-coordinate of its centre of gravity is given by the formula

$$z_C = \frac{\iiint\limits_V z\gamma_0\,dx\,dy\,dz}{\iiint\limits_V \gamma_0\,dx\,dy\,dz}$$

Changing to spherical coordinates, we get

$$z_C = \frac{\gamma_0 \int\limits_0^{2\pi} \left[ \int\limits_0^{\frac{\pi}{2}} \left( \int\limits_0^R r\cos\varphi r^2 \sin\varphi\,dr \right) d\varphi \right] d\theta}{\gamma_0 \int\limits_0^{2\pi} \left[ \int\limits_0^{\frac{\pi}{2}} \left( \int\limits_0^R r^2 \sin\varphi\,dr \right) d\varphi \right] d\theta} = \frac{2\pi \frac{R^4}{4} \frac{1}{2}}{\frac{4}{6}\pi R^3} = \frac{3}{8} R$$

Obviously, by virtue of the symmetry of the hemisphere, $x_c = y_c = 0$.

## 2.15 COMPUTING INTEGRALS DEPENDENT ON A PARAMETER

Consider an integral dependent on the parameter $\alpha$:

$$I(\alpha) = \int\limits_a^b f(x, \alpha)\,dx$$

(We examined such integrals in Sec. 11.10, Vol. I.) We state without proof that if a function $f(x, \alpha)$ is continuous with respect to

$x$ over an interval $[a, b]$ and with respect to $\alpha$ over an interval $[\alpha_1, \alpha_2]$, then the function

$$I(\alpha)=\int_a^b f(x, \alpha)\,dx$$

is a continuous function on $[\alpha_1, \alpha_2]$. Consequently, the function $I(\alpha)$ may be integrated with respect to $\alpha$ over the interval $[\alpha_1, \alpha_2]$:

$$\int_{\alpha_1}^{\alpha_2} I(\alpha)\,d\alpha=\int_{\alpha_1}^{\alpha_2}\left[\int_a^b f(x, \alpha)\,dx\right]d\alpha$$

The expression on the right is an iterated integral of the function $f(x, \alpha)$ over a rectangle situated in the plane $xO\alpha$. We can change the order of integration in this integral:

$$\int_{\alpha_1}^{\alpha_2}\left[\int_a^b f(x, \alpha)\,dx\right]d\alpha=\int_a^b\left[\int_{\alpha_1}^{\alpha_2} f(x, \alpha)\,d\alpha\right]dx$$

This formula shows that for integration of an integral dependent on a parameter $\alpha$, it is sufficient to integrate the element of integration with respect to the parameter $\alpha$. This formula is also useful when computing definite integrals.

**Example.** Compute the integral

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx \quad (a>0,\ b>0)$$

This integral is not expressible in terms of elementary functions. To evaluate it, we consider another integral that may be readily computed:

$$\int_0^\infty e^{-\alpha x}\,dx=\frac{1}{\alpha} \quad (\alpha>0)$$

Integrating this equation between the limits $\alpha=a$ and $\alpha=b$, we get

$$\int_a^b\left[\int_0^\infty e^{-\alpha x}\,dx\right]d\alpha=\int_a^b\frac{d\alpha}{\alpha}=\ln\frac{b}{a}$$

Changing the order of integration in the first integral, we rewrite this equation in the following form:

$$\int_0^\infty\left[\int_a^b e^{-\alpha x}\,d\alpha\right]dx=\ln\frac{b}{a}$$

whence, computing the inner integral, we get

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx=\ln\frac{b}{a}$$

## Exercises on Chapter 2

Evaluate the integrals: *

1. $\int_0^1\int_1^2 (x^2+y^2)\,dx\,dy$. *Ans.* $\frac{8}{3}$. 2. $\int_3^4\int_1^2 \frac{dy\,dx}{(x+y)^2}$.* *Ans.* $\ln\frac{25}{24}$. 3. $\int_1^2\int_x^{x\sqrt{3}} xy\,dx\,dy$. *Ans.* $\frac{15}{4}$. 4. $\int_0^{2\pi}\int_{a\sin\theta}^{a} r\,dr\,d\theta$. *Ans.* $\frac{1}{2}\pi a^2$. 5. $\int_0^a\int_{\frac{x}{a}}^{x} \frac{x\,dy\,dx}{x^2+y^2}$. *Ans.* $\frac{\pi a}{4}-a\arctan\frac{1}{a}$.

6. $\int_0^a\int_{y-a}^{2y} xy\,dx\,dy$. *Ans.* $\frac{11a^4}{24}$. 7. $\int_{\frac{b}{2}}^{b}\int_0^{\pi/2} \rho\,d\theta\,d\rho$. *Ans.* $\frac{3}{16}\pi b^2$.

Determine the limits of integration for the integral $\iint_D f(x, y)\,dx\,dy$ where the domain of integration is bounded by the lines:

8. $x=2$, $x=3$, $y=-1$, $y=5$. *Ans.* $\int_2^3\int_{-1}^5 f(x, y)\,dy\,dx$. 9. $y=0$, $y=1-x^2$. *Ans.* $\int_{-1}^1\int_0^{1-x^2} f(x, y)\,dy\,dx$. 10. $x^2+y^2=a^2$. *Ans.* $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\,dy\,dx$.

11. $y=\frac{2}{1+x^2}$, $y=x^2$. *Ans.* $\int_{-1}^{1}\int_{x^2}^{\frac{2}{1+x^2}} f(x, y)\,dy\,dx$. 12. $y=0$, $y=a$, $y=x$, $y=x-2a$. *Ans.* $\int_0^a\int_y^{y+2a} f(x, y)\,dx\,dy$.

Change the order of integration in the integrals:

13. $\int_1^2\int_3^4 f(x, y)\,dy\,dx$. *Ans.* $\int_3^4\int_1^2 f(x, y)\,dx\,dy$. 14. $\int_0^1\int_{x^3}^{\sqrt{x}} f(x, y)\,dy\,dx$. *Ans.* $\int_0^1\int_{y^2}^{\sqrt[3]{y}} f(x, y)\,dx\,dy$. 15. $\int_0^a\int_0^{\sqrt{2ay-y^2}} f(x, y)\,dx\,dy$. *Ans.* $\int_0^a\int_{a-\sqrt{a^2-x^2}}^{a} f(x, y)\,dy\,dx$.

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* If the integral is written as $\int_M^N\int_K^L f(x, y)\,dx\,dy$ then, as has already been stated, we can consider that the first integration is performed with respect to the variable whose differential occupies the first place; that is,

$$\int_M^N\int_K^L f(x, y)\,dx\,dy=\int_M^N\left(\int_K^L f(x, y)\,dx\right)dy$$

16. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} f(x, y)\,dy\,dx.$ *Ans.* $\int_{0}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y)\,dx\,dy.$

17. $\int_{0}^{1}\int_{-\sqrt{1-y^2}}^{1-y} f(x, y)\,dx\,dy.$ *Ans.* $\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} f(x, y)\,dy\,dx+\int_{0}^{1}\int_{0}^{1-x} f(x, y)\,dy\,dx.$

Compute the following integrals by changing to polar coordinates:

18. $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx.$ *Ans.* $\int_{0}^{\frac{\pi}{2}}\int_{0}^{a} \sqrt{a^2-\rho^2}\,\rho\,d\rho\,d\theta=\frac{\pi}{6}a^3.$

19. $\int_{0}^{a}\int_{0}^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy.$ *Ans.* $\int_{0}^{\frac{\pi}{2}}\int_{0}^{a} \rho^3\,d\rho\,d\theta=\frac{\pi a^4}{8}.$ 20. $\int_{0}^{\infty}\int_{0}^{\infty} e^{-(x^2+y^2)}dy\,dx.$

*Ans.* $\int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty} e^{-\rho^2}\rho\,d\rho\,d\theta=\frac{\pi}{4}.$ 21. $\int_{0}^{2a}\int_{0}^{\sqrt{2ax-x^2}} dy\,dx.$ *Ans.* $\int_{0}^{\frac{\pi}{2}}\int_{0}^{2a\cos\theta} \rho\,d\rho\,d\theta=\frac{\pi a^2}{2}.$

Transform the double integrals by introducing new variables $u$ and $v$ connected with $x$ and $y$ by the formulas $x=u-uv,\ y=uv$:

22. $\int_{0}^{e}\int_{\alpha x}^{\beta x} f(x, y)\,dy\,dx.$ *Ans.* $\int_{\frac{\alpha}{1+\alpha}}^{\frac{\beta}{1+\beta}}\int_{0}^{\frac{e}{1+v}} f(u-uv,\ uv)\,u\,du\,dv.$ 23. $\int_{0}^{c}\int_{0}^{b} f(x, y)\,dy\,dx.$

*Ans.* $\int_{0}^{\frac{b}{b+c}}\int_{0}^{\frac{c}{1-v}} f(u-uv,\ uv)\,u\,du\,dv+\int_{\frac{b}{b+c}}^{b}\int_{0}^{\frac{b}{v}} f(u-uv,\ uv)\,u\,du\,dv.$

## Calculating Areas by Means of Double Integrals

24. Compute the area of a figure bounded by the parabola $y^2=2x$ and the straight line $y=x$. *Ans.* $\frac{2}{3}$.

25. Compute the area of a figure bounded by the curves $y^2=4ax$, $x+y=3a$, $y=0$. *Ans.* $\frac{10}{3}a^2$.

26. Compute the area of a figure bounded by the curves $x^{1/2}+y^{1/2}=a^{1/2}$, $x+y=a$. *Ans.* $\frac{a^2}{3}$.

27. Compute the area of a figure bounded by the curves $y=\sin x$, $y=\cos x$, $x=0$. *Ans.* $\sqrt{2}-1$.

28. Compute the area of a loop of the curve $\rho=a\sin 2\theta$. *Ans.* $\frac{\pi a^2}{8}$.

29. Compute the entire area bounded by the lemniscate $\rho^2=a^2\cos 2\varphi$. *Ans.* $a^2$.