THE INDEFINITE INTEGRAL

10.1 ANTIDERIVATIVE AND THE INDEFINITE INTEGRAL

In Chapter 3 we considered the following problem: given a function $F(x)$, find its derivative, that is, the function $f(x)=F'(x)$.

In this chapter we shall consider the reverse problem: given a function $f(x)$, it is required to find a function $F(x)$ such that its derivative is equal to $f(x)$, that is,

$$F'(x)=f(x)$$

Definition 1. A function $F(x)$ is called the *antiderivative* of the function $f(x)$ on the interval $[a, b]$ if at all points of the interval $F'(x)=f(x)$.

Example. Find the antiderivative of the function $f(x)=x^2$.

From the definition of an antiderivative it follows that the function $F(x)=\frac{x^3}{3}$ is an antiderivative, since $\left(\frac{x^3}{3}\right)'=x^2$.

It is easy to see that if for the given function $f(x)$ there exists an antiderivative, then this antiderivative is not the only one. In the foregoing example, we could take the following functions as antiderivatives: $F(x)=\frac{x^3}{3}+1$, $F(x)=\frac{x^3}{3}-7$ or, generally, $F(x)=\frac{x^3}{3}+C$ (where C is an arbitrary constant), since

$$\left(\frac{x^3}{3}+C\right)'=x^2$$

On the other hand, it may be proved that functions of the form $\frac{x^3}{3}+C$ exhaust all antiderivatives of the function x^2. This is a consequence of the following theorem.

Theorem. *If $F_1(x)$ and $F_2(x)$ are two antiderivatives of a function $f(x)$ on an interval $[a, b]$, then the difference between them is a constant.*

Proof. By virtue of the definition of an antiderivative we have

$$\left.\begin{aligned} F_1'(x)&=f(x)\\ F_2'(x)&=f(x)\end{aligned}\right\} \qquad (1)$$

for any value of x on the interval $[a, b]$.

Let us put

$$F_1(x)-F_2(x)=\varphi(x) \tag{2}$$

Then by (1) we have

$$F_1'(x)-F_2'(x)=f(x)-f(x)=0$$

or

$$\varphi'(x)=[F_1(x)-F_2(x)]'\equiv 0$$

for any value of x on the interval $[a, b]$. But from $\varphi'(x)=0$ it follows that $\varphi(x)$ is a constant.

Indeed, let us apply the Lagrange theorem (see Sec. 4.2) to the function $\varphi(x)$, which, obviously, is continuous and differentiable on the interval $[a, b]$. No matter what the point x on the interval $[a, b]$, we have, by virtue of the Lagrange theorem,

$$\varphi(x)-\varphi(a)=(x-a)\varphi'(\xi)$$

where $a<\xi<x$.

Since $\varphi'(\xi)=0$,

$$\varphi(x)-\varphi(a)=0$$

or

$$\varphi(x)=\varphi(a) \tag{3}$$

Thus, the function $\varphi(x)$ at any point x of the interval $[a, b]$ retains the value $\varphi(a)$, and this means that the function $\varphi(x)$ is constant on $[a, b]$. Denoting the constant $\varphi(a)$ by C, we get, from (2) and (3),

$$F_1(x)-F_2(x)=C$$

From this theorem it follows that if for a given function $f(x)$ **some** one antiderivative $F(x)$ is found, then **any other** antiderivative of $f(x)$ has the form $F(x)+C$, where $C=\text{constant}$.

Definition 2. If the function $F(x)$ is an antiderivative of $f(x)$, then the expression $F(x)+C$ is the *indefinite integral* of the function $f(x)$ and is denoted by the symbol $\int f(x)\,dx$. Thus, by definition

$$\int f(x)\,dx=F(x)+C$$

if

$$F'(x)=f(x)$$

Here, the function $f(x)$ is called the *integrand*, $f(x)\,dx$ is the *element of integration* (the expression under the integral sign), and $\int$ is the *integral sign*.

Thus, an indefinite integral is a **family of functions** $y=F(x)+C$.

From the geometrical point of view, an indefinite integral is a collection (family) of curves, each of which is obtained by translating one of the curves parallel to itself upwards or downwards (that is, along the y-axis).

A natural question arises: do antiderivatives (and, hence, indefinite integrals) exist for every function $f(x)$? The answer is no. Let us note, however, without proof, that *if a function $f(x)$ is continuous on an interval $[a, b]$, then this function has an antiderivative* (and, hence, *there is also an indefinite integral*).

This chapter is devoted to working out methods by means of which we can find antiderivatives (and indefinite integrals) for certain classes of elementary functions.

The finding of an antiderivative of a given function $f(x)$ is called *integration* of the function $f(x)$.

Note the following: if the derivative of an elementary function is always an elementary function, then the antiderivative of the elementary function may not prove to be representable by a finite number of elementary functions. We shall return to this question at the end of the chapter.

From Definition 2 it follows that:

1. *The derivative of an indefinite integral is equal to the integrand, that is, if $F'(x) = f(x)$, then also*

$$\left(\int f(x)dx\right)' = (F(x) + C)' = f(x). \tag{4}$$

This equation should be understood in the sense that the derivative of any antiderivative is equal to the integrand.

2. *The differential of an indefinite integral is equal to the expression under the integral sign:*

$$d\left(\int f(x)\,dx\right) = f(x)\,dx \tag{5}$$

This results from formula (4).

3. *The indefinite integral of the differential of some function is equal to this function plus an arbitrary constant:*

$$\int dF(x) = F(x) + C$$

The truth of this equation may easily be checked by differentiation [the differentials of both sides are equal to $dF(x)$].

10.2 TABLE OF INTEGRALS

Before starting on methods of integration, we give the following table of integrals of the simplest functions.

The table of integrals follows directly from Definition 2, Sec. 10.1, and from the table of derivatives (given in Sec. 3.15). (The truth of the equations can easily be checked by differentiation: by establishing that the derivative of the right side is equal to the integrand.)

1. $\int x^{\alpha}\,dx=\frac{x^{\alpha+1}}{\alpha+1}+C$ $(\alpha\neq-1)$. (Here and in the formulas that follow, C stands for an arbitrary constant.)

2. $\int \frac{dx}{x}=\ln|x|+C.$

3. $\int \sin x\,dx=-\cos x+C.$

4. $\int \cos x\,dx=\sin x+C.$

5. $\int \frac{dx}{\cos^2 x}=\tan x+C.$

6. $\int \frac{dx}{\sin^2 x}=-\cot x+C.$

7. $\int \tan x\,dx=-\ln|\cos x|+C.$

8. $\int \cot x\,dx=\ln|\sin x|+C.$

9. $\int e^x\,dx=e^x+C.$

10. $\int a^x\,dx=\frac{a^x}{\ln a}+C.$

11. $\int \frac{dx}{1+x^2}=\arctan x+C.$

11′. $\int \frac{dx}{a^2+x^2}=\frac{1}{a}\arctan\frac{x}{a}+C.$

12. $\int \frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C.$

13. $\int \frac{dx}{\sqrt{1-x^2}}=\arcsin x+C.$

13′. $\int \frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+C.$

14. $\int \frac{dx}{\sqrt{x^2\pm a^2}}=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C.$

Note. The table of derivatives (Sec. 3.15) does not have formulas corresponding to formulas 7, 8, 11′, 12, 13′ and 14. However, differentiation will readily prove the truth of these as well.

In the case of Formula 7 we have

$$(-\ln|\cos x|)'=-\frac{-\sin x}{\cos x}=\tan x$$

consequently, $\int \tan x\,dx=-\ln|\cos x|+C.$

In the case of Formula 8

$$(\ln|\sin x|)' = \frac{\cos x}{\sin x} = \cot x$$

Consequently, $\int \cot x\,dx = \ln|\sin x| + C$.

In the case of Formula 12,

$$\left(\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|\right)' = \frac{1}{2a}[\ln|a+x| - \ln|a-x|]'$$
$$= \frac{1}{2a}\left[\frac{1}{a+x} + \frac{1}{a-x}\right] = \frac{1}{a^2-x^2}$$

therefore,

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

It should be noted that the latter formula will also follow from the general results of Sec. 10.9.

In the case of Formula 14,

$$(\ln|x+\sqrt{x^2 \pm a^2}|)' = \frac{1}{x+\sqrt{x^2 \pm a^2}}\left(1 + \frac{x}{\sqrt{x^2 \pm a^2}}\right) = \frac{1}{\sqrt{x^2 \pm a^2}}$$

hence,

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x+\sqrt{x^2 \pm a^2}| + C$$

This formula likewise will follow from the general results of Sec. 10.10.

Formulas 11′ and 13′ may be verified in similar fashion. These formulas will later be derived from formulas 11 and 13 (see Sec. 10.4, Examples 3 and 4).

10.3 SOME PROPERTIES OF THE INDEFINITE INTEGRAL

Theorem 1. *The indefinite integral of an algebraic sum of two or more functions is equal to the algebraic sum of their integrals*

$$\int [f_1(x) + f_2(x)]\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx \qquad (1)$$

To prove this, find the derivatives of the left and right sides of this equation. On the basis of (4) of the preceding section we have

$$\left(\int [f_1(x) + f_2(x)]\,dx\right)' = f_1(x) + f_2(x)$$

$$\left(\int f_1(x)\,dx + \int f_2(x)\,dx\right)'$$
$$= \left(\int f_1(x)\,dx\right)' + \left(\int f_2(x)\,dx\right)' = f_1(x) + f_2(x)$$

Thus, the derivatives of the left and right sides of (1) are equal; in other words, the derivative of any antiderivative on the left-hand side is equal to the derivative of any function on the right-hand side of the equation. Therefore, by the theorem of Sec. 10.1, any function on the left of (1) differs from any function on the right of (1) by a constant term. That is how we should understand equation (1).

Theorem 2. *A constant factor may be taken outside the integral sign; that is, if $a = const$, then*

$$\int af(x)\,dx = a\int f(x)\,dx \tag{2}$$

To prove (2), let us find the derivatives of the left and right sides:

$$\left(\int af(x)\,dx\right)' = af(x)$$

$$\left(a\int f(x)dx\right)' = a\left(\int f(x)\,dx\right)' = af(x)$$

The derivatives of the right and left sides are equal, therefore, as in (1), the difference of any two functions on the left and right is a constant. That is how we should understand equation (2).

When evaluating indefinite integrals it is useful to bear in mind the following rules.

I. If

$$\int f(x)\,dx = F(x) + C$$

then

$$\int f(ax)\,dx = \frac{1}{a}F(ax) + C \tag{3}$$

Indeed, differentiating the left and right sides of (3), we get

$$\left(\int f(ax)\,dx\right)' = f(ax)$$

$$\left(\frac{1}{a}F(ax)\right)' = \frac{1}{a}(F(ax))'_x = \frac{1}{a}F'(ax)\,a = F'(ax) = f(ax)$$

The derivatives of the right and left sides are equal, which is what we set out to prove.

II. If

$$\int f(x)\,dx = F(x) + C$$

then

$$\int f(x+b)\,dx = F(x+b) + C \tag{4}$$

III. If

$$\int f(x)\,dx = F(x) + C$$

then

$$\int f(ax+b)\,dx=\frac{1}{a}F(ax+b)+C \tag{5}$$

Equations (4) and (5) are proved by differentiation of the right and left sides.

Example 1.

$$\int (2x^3-3\sin x+5\sqrt{x})\,dx=\int 2x^3\,dx-\int 3\sin x\,dx+\int 5\sqrt{x}\,dx$$

$$=2\int x^3\,dx-3\int \sin x\,dx+5\int x^{\frac{1}{2}}\,dx$$

$$=2\frac{x^{3+1}}{3+1}-3(-\cos x)+5\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C=\frac{1}{2}x^4+3\cos x+\frac{10}{3}x\sqrt{x}+C$$

Example 2.

$$\int\left(\frac{3}{\sqrt[3]{x}}+\frac{1}{2\sqrt{x}}+x\sqrt[4]{x}\right)dx=3\int x^{-\frac{1}{3}}\,dx+\frac{1}{2}\int x^{-\frac{1}{2}}\,dx+\int x^{\frac{5}{4}}\,dx$$

$$=3\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+\frac{1}{2}\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+\frac{x^{\frac{5}{4}+1}}{\frac{5}{4}+1}+C=\frac{9}{2}\sqrt[3]{x^2}+\sqrt{x}+\frac{4}{9}x^2\sqrt[4]{x}+C$$

Example 3.

$$\int\frac{dx}{x+3}=\ln|x+3|+C$$

Example 4.

$$\int \cos 7x\,dx=\frac{1}{7}\sin 7x+C$$

Example 5.

$$\int \sin(2x-6)\,dx=-\frac{1}{2}\cos(2x-6)+C$$

10.4 INTEGRATION BY SUBSTITUTION (CHANGE OF VARIABLE)

Let it be required to find the integral

$$\int f(x)\,dx;$$

we cannot directly select the antiderivative of $f(x)$ but we know that it exists.

Let us change the variable in the expression under the integral sign, putting

$$x=\varphi(t) \tag{1}$$

where $\varphi(t)$ is a continuous function (with continuous derivative) having an inverse function. Then $dx = \varphi'(t)\,dt$; we shall prove that in this case we have the following equation:

$$\int f(x)\,dx = \int f[\varphi(t)]\,\varphi'(t)\,dt \tag{2}$$

Here it is assumed that after integration we substitute, on the right side, the expression of t in terms of x on the basis of (1).

To establish that the expressions to the right and left are the same in the sense indicated above, it is necessary to prove that their derivatives with respect to x are equal. Find the derivative of the left side:

$$\left(\int f(x)\,dx\right)'_x = f(x)$$

We differentiate the right side of (2) with respect to x as a composite function, where t is the intermediate argument. The dependence of t on x is expressed by (1); here, $\frac{dx}{dt} = \varphi'(t)$ and by the rule for differentiating an inverse function,

$$\frac{dt}{dx} = \frac{1}{\varphi'(t)}$$

We thus have

$$\left(\int f[\varphi(t)]\,\varphi'(t)\,dt\right)'_x = \left(\int f[\varphi(t)]\,\varphi'(t)\,dt\right)'_t \frac{dt}{dx}$$

$$= f[\varphi(t)]\,\varphi'(t)\,\frac{1}{\varphi'(t)} = f[\varphi(t)] = f(x)$$

Therefore, the derivatives, with respect to x, of the right and left sides of (2) are equal, as required.

The function $x = \varphi(t)$ should be chosen so that one can evaluate the indefinite integral on the right side of (2).

Note. When integrating, it is sometimes better to choose a change of the variable in the form of $t = \psi(x)$ and not $x = \varphi(t)$. By way of illustration, let it be required to calculate an integral of the form

$$\int \frac{\psi'(x)\,dx}{\psi(x)}$$

Here it is convenient to put

$$\psi(x) = t$$

then

$$\psi'(x)\,dx = dt$$

$$\int \frac{\psi'(x)\,dx}{\psi(x)} = \int \frac{dt}{t} = \ln|t| + C = \ln|\psi(x)| + C$$

The following are some instances of integration by substitution.

Example 1. $\int \sqrt{\sin x}\cos x\,dx = ?$ We make the substitution $t=\sin x$; then $dt=\cos x\,dx$ and, consequently, $\int \sqrt{\sin x}\cos x\,dx=\int \sqrt{t}\,dt=\int t^{\frac{1}{2}}\,dt = \frac{2t^{3/2}}{3}+C=\frac{2}{3}\sin^{3/2}x+C.$

Example 2. $\int \frac{x\,dx}{1+x^2}=?$ We put $t=1+x^2$, then $dt=2x\,dx$ and $\int\frac{x\,dx}{1+x^2}= =\frac{1}{2}\int\frac{dt}{t}=\frac{1}{2}\ln t+C=\frac{1}{2}\ln(1+x^2)+C.$

Example 3. $\int\frac{dx}{a^2+x^2}=\frac{1}{a^2}\int\frac{dx}{1+\left(\frac{x}{a}\right)^2}$. We put $t=\frac{x}{a}$; then $dx=a\,dt$,

$\int\frac{dx}{a^2+x^2}=\frac{1}{a^2}\int\frac{a\,dt}{1+t^2}=\frac{1}{a}\int\frac{dt}{1+t^2}=\frac{1}{a}\arctan t+C=\frac{1}{a}\arctan\frac{x}{a}+C.$

Example 4. $\int\frac{dx}{\sqrt{a^2-x^2}}=\frac{1}{a}\int\frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$. We put $t=\frac{x}{a}$; then $dx=a\,dt$, $\int\frac{dx}{\sqrt{a^2-x^2}}=\frac{1}{a}\int\frac{a\,dt}{\sqrt{1-t^2}}=\int\frac{dt}{\sqrt{1-t^2}}=\arcsin t+C= =\arcsin\frac{x}{a}+C$ (it is assumed that $a>0$).

Examples 3 and 4 illustrate the derivation of formulas 11′ and 13′ given in the Table of Integrals (see above, Sec. 10.2).

Example 5. $\int(\ln x)^3\frac{dx}{x}=?$ Put $t=\ln x$; then $dt=\frac{dx}{x}$, $\int(\ln x)^3\frac{dx}{x}= =\int t^3\,dt=\frac{t^4}{4}+C=\frac{1}{4}(\ln x)^4+C.$

Example 6. $\int\frac{x\,dx}{1+x^4}=?$ Put $t=x^2$; then $dt=2x\,dx$, $\int\frac{x\,dx}{1+x^4}=\frac{1}{2}\int\frac{dt}{1+t^2}= =\frac{1}{2}\arctan t+C=\frac{1}{2}\arctan x^2+C$

The method of substitution is one of the basic methods for calculating indefinite integrals. Even when we integrate by some other method, we often resort to substitution in the intermediate stages of calculation. The success of integration depends largely on how appropriate the substitution is for simplifying the given integral. Essentially, the study of methods of integration reduces to finding out what kind of substitution has to be performed for a given element of integration. Most of this chapter is devoted to this problem.

10.5 INTEGRALS OF SOME FUNCTIONS CONTAINING A QUADRATIC TRINOMIAL

I. Let us consider the integral

$$I_1=\int\frac{dx}{ax^2+bx+c}$$

We first transform the trinomial in the denominator by representing it in the form of a sum or difference of squares:

$$ax^2+bx+c=a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$$
$$=a\left[x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right]$$
$$=a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]=a\left[\left(x+\frac{b}{2a}\right)^2\pm k^2\right]$$

where

$$\frac{c}{a}-\frac{b^2}{4a^2}=\pm k^2$$

The plus or minus sign is taken depending on whether the expression on the left is positive or negative, that is, on whether the roots of the trinomial ax^2+bx+c are complex or real.

Thus, the integral I_1 will take the form

$$I_1=\int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{dx}{\left[\left(x+\frac{b}{2a}\right)^2\pm k^2\right]}$$

In this integral we make a change of variable:

$$x+\frac{b}{2a}=t,\quad dx=dt$$

We then get

$$I_1=\frac{1}{a}\int\frac{dt}{t^2\pm k^2}$$

These are tabular integrals (see Formulas 11′ and 12).

Example 1. Calculate the integral

$$\int\frac{dx}{2x^2+8x+20}$$

Solution.

$$I=\int\frac{dx}{2x^2+8x+20}=\frac{1}{2}\int\frac{dx}{x^2+4x+10}$$
$$=\frac{1}{2}\int\frac{dx}{x^2+4x+4+10-4}=\frac{1}{2}\int\frac{dx}{(x+2)^2+6}$$

Let us make the substitution $x+2=t$, $dx=dt$. Putting it into the integral, we get the tabular integral

$$I=\frac{1}{2}\int\frac{dt}{t^2+6}=\frac{1}{2}\frac{1}{\sqrt{6}}\arctan\frac{t}{\sqrt{6}}+C$$

Substituting in place of t its expression in terms of x, we finally get

$$I=\frac{1}{2\sqrt{6}}\arctan\frac{x+2}{\sqrt{6}}+C.$$

II. Let us consider an integral of a more general form:

$$I_2=\int\frac{Ax+B}{ax^2+bx+c}dx$$

Perform the identity transformation of the integrand:

$$I_2=\int\frac{Ax+B}{ax^2+bx+c}dx=\int\frac{\frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right)}{ax^2+bx+c}dx$$

Represent the latter integral in the form of a sum of two integrals. Taking the constant factors outside the integral sign, we get

$$I_2=\frac{A}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx+\left(B-\frac{Ab}{2a}\right)\int\frac{dx}{ax^2+bx+c}$$

The latter integral is the integral I_1, which we are able to evaluate. In the first integral make a change of variable:

$$ax^2+bx+c=t,\quad (2ax+b)\,dx=dt$$

Thus,

$$\int\frac{(2ax+b)\,dx}{ax^2+bx+c}=\int\frac{dt}{t}=\ln|t|+C=\ln|ax^2+bx+c|+C$$

And we finally get

$$I_2=\frac{A}{2a}\ln|ax^2+bx+c|+\left(B-\frac{Ab}{2a}\right)I_1$$

Example 2. Evaluate the integral

$$I=\int\frac{x+3}{x^2-2x-5}dx.$$

Applying the foregoing technique we have

$$I=\int\frac{x+3}{x^2-2x-5}dx=\int\frac{\frac{1}{2}(2x-2)+\left(3+\frac{1}{2}2\right)}{x^2-2x-5}dx$$

$$=\frac{1}{2}\int\frac{(2x-2)\,dx}{x^2-2x-5}+4\int\frac{dx}{x^2-2x-5}$$

$$=\frac{1}{2}\ln|x^2-2x-5|+4\int\frac{dx}{(x-1)^2-6}$$

$$=\frac{1}{2}\ln|x^2-2x-5|+2\frac{1}{\sqrt{6}}\ln\left|\frac{\sqrt{6}-(x-1)}{\sqrt{6}+(x-1)}\right|+C$$

III. Let us consider the integral

$$\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

By means of transformations considered in Item I, this integral reduces (depending on the sign of a) to tabular integrals of the form

$$\int \frac{dt}{\sqrt{t^2 \pm k^2}} \text{ for } a>0 \text{ or } \int \frac{dt}{\sqrt{k^2-t^2}} \text{ for } a<0$$

which have already been examined in the Table of Integrals (see formulas 13′ and 14).

IV. An integral of the form

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$$

is evaluated by means of the following transformations, which are similar to those considered in Item II:

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx = \int \frac{\frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right)}{\sqrt{ax^2+bx+c}}\,dx$$
$$=\frac{A}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx+\left(B-\frac{Ab}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

Applying substitution to the first of the integrals obtained,

$$ax^2+bx+c=t, \quad (2ax+b)\,dx=dt$$

we get

$$\int \frac{(2ax+b)\,dx}{\sqrt{ax^2+bx+c}}=\int \frac{dt}{\sqrt{t}}=2\sqrt{t}+C=2\sqrt{ax^2+bx+c}+C$$

The second integral was considered in Item III of this section.

Example 3.

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx=\int \frac{\frac{5}{2}(2x+4)+(3-10)}{\sqrt{x^2+4x+10}}\,dx$$
$$=\frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}\,dx-7\int \frac{dx}{\sqrt{(x+2)^2+6}}$$
$$=5\sqrt{x^2+4x+10}-7\ln\left|x+2+\sqrt{(x+2)^2+6}\right|+C$$
$$=5\sqrt{x^2+4x+10}-7\ln\left|x+2+\sqrt{x^2+4x+10}\right|+C$$

10.6 INTEGRATION BY PARTS

Let u and v be two differentiable functions of x. Then the differential of the product uv is found from the following formula:

$$d(uv)=u\,dv+v\,du$$

Whence, by integration, we have

$$uv = \int u\,dv + \int v\,du$$

or

$$\int u\,dv = uv - \int v\,du \qquad (1)$$

This formula is called the *formula of integration by parts*. It is most frequently used in the integration of expressions that may be represented in the form of a product of two factors u and dv in such a way that the finding of the function v from its differential dv, and the evaluation of the integral $\int v\,du$ should, taken together, be a simpler problem than the direct evaluation of the integral $\int u\,dv$. To become skilled at breaking up a given element of integration into the factors u and dv, one has to solve problems; we shall show how this is done in a number of cases.

Example 1. $\int x \sin x\,dx = ?$ We let

$$u = x, \quad dv = \sin x\,dx$$

then

$$du = dx, \quad v = -\cos x$$

Hence,

$$\int x \sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

Note. When determining the function v from the differential dv we can take any arbitrary constant, since it does not enter into the final result [this can be seen by putting the expression $v+C$ into (1) in place of v]. It is therefore convenient to consider this constant equal to zero.

The rule for integration by parts is widely used. For example, integrals of the form

$$\int x^k \sin ax\,dx, \qquad \int x^k \cos ax\,dx$$

$$\int x^k e^{ax}\,dx, \qquad \int x^k \ln x\,dx$$

and certain integrals containing inverse trigonometric functions are evaluated by means of integration by parts.

Example 2. It is required to evaluate $\int \arctan x\,dx$. Letting $u = \arctan x$, $dv = dx$, we have $du = \frac{dx}{1+x^2}$, $v = x$. Thus,

$$\int \arctan x\,dx = x\arctan x - \int \frac{x\,dx}{1+x^2} = x\arctan x - \frac{1}{2}\ln|1+x^2| + C$$

Example 3. It is required to evaluate $\int x^2 e^x \, dx$. Let us put $u = x^2$, $dv = e^x \, dx$; then $du = 2x \, dx$, $v = e^x$,

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx$$

We again integrate by parts the latter integral, letting

$$u_1 = x, \qquad du_1 = dx$$
$$dv_1 = e^x \, dx, \qquad v_1 = e^x$$

Then

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$$

Finally we get

$$\int x^2 e^x \, dx = x^2 e^x - 2(x e^x - e^x) + C = x^2 e^x - 2x e^x + 2e^x + C = e^x (x^2 - 2x + 2) + C$$

Example 4. It is required to evaluate $\int (x^2 + 7x - 5) \cos 2x \, dx$. We let $u = x^2 + 7x - 5$; $dv = \cos 2x \, dx$; then

$$du = (2x + 7) \, dx, \qquad v = \frac{\sin 2x}{2}$$

$$\int (x^2 + 7x - 5) \cos 2x \, dx = (x^2 + 7x - 5) \frac{\sin 2x}{2} - \int (2x + 7) \frac{\sin 2x}{2} \, dx$$

Apply integration by parts to the latter integral, letting $u_1 = \frac{2x+7}{2}$, $dv_1 =$ $= \sin 2x \, dx$; then

$$du_1 = dx, \quad v_1 = -\frac{\cos 2x}{2}$$

$$\int \frac{2x+7}{2} \sin 2x \, dx = \frac{2x+7}{2} \left(-\frac{\cos 2x}{2} \right) - \int \left(-\frac{\cos 2x}{2} \right) dx$$
$$= -\frac{(2x+7) \cos 2x}{4} + \frac{\sin 2x}{4} + C$$

Therefore, we finally get

$$\int (x^2 + 7x - 5) \cos 2x \, dx = (x^2 + 7x - 5) \frac{\sin 2x}{2} + (2x + 7) \frac{\cos 2x}{4} - \frac{\sin 2x}{4} + C$$
$$= (2x^2 + 14x - 11) \frac{\sin x}{4} + (2x + 7) \frac{\cos 2x}{4} + C$$

Example 5. $I = \int \sqrt{a^2 - x^2} \, dx = ?$

Perform identity transformations. Multiply and divide the integrand by $\sqrt{a^2 - x^2}$:

$$\int \sqrt{a^2 - x^2} \, dx = \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} \, dx = a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - \int \frac{x^2 \, dx}{\sqrt{a^2 - x^2}}$$
$$= a^2 \arcsin \frac{x}{a} - \int x \frac{x \, dx}{\sqrt{a^2 - x^2}}$$

Integrate the last integral by parts, letting

$$u=x, \qquad du=dx$$

$$dv=\frac{x\,dx}{\sqrt{a^2-x^2}}, \quad v=-\sqrt{a^2-x^2}$$

Then

$$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}=\int x\frac{x\,dx}{\sqrt{a^2-x^2}}=-x\sqrt{a^2-x^2}+\int \sqrt{a^2-x^2}\,dx$$

Putting this result in the earlier obtained expression of the given integral, we have

$$\int \sqrt{a^2-x^2}\,dx=a^2\arcsin\frac{x}{a}+x\sqrt{a^2-x^2}-\int \sqrt{a^2-x^2}\,dx$$

Transposing the integral from right to left and performing elementary transformations, we finally get

$$\int \sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C$$

Example 6. Evaluate the integrals

$$I_1=\int e^{ax}\cos bx\,dx \quad \text{and} \quad I_2=\int e^{ax}\sin bx\,dx$$

Applying integration by parts to the first integral, we get

$$u=e^{ax}, \qquad du=ae^{ax}\,dx$$

$$dv=\cos bx\,dx, \quad v=\frac{1}{b}\sin bx$$

$$\int e^{ax}\cos bx\,dx=\frac{1}{b}e^{ax}\sin bx-\frac{a}{b}\int e^{ax}\sin bx\,dx$$

Again apply the method of integration by parts to the latter integral:

$$u=e^{ax}, \qquad du=ae^{ax}\,dx$$

$$dv=\sin bx\,dx, \quad v=-\frac{1}{b}\cos bx$$

$$\int e^{ax}\sin bx\,dx=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b}\int e^{ax}\cos bx\,dx$$

Putting the expression obtained into the preceding equation gives us

$$\int e^{ax}\cos bx\,dx=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}e^{ax}\cos bx-\frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

From this equation let us find I_1

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx=e^{ax}\left(\frac{1}{b}\sin bx+\frac{a}{b^2}\cos bx\right)+C\left(1+\frac{a^2}{b^2}\right)$$

whence

$$I_1=\int e^{ax}\cos bx\,dx=\frac{e^{ax}(b\sin bx+a\cos bx)}{a^2+b^2}+C$$

Similarly we find

$$I_2=\int e^{ax}\sin bx\,dx=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}+C$$

10.7 RATIONAL FRACTIONS. PARTIAL RATIONAL FRACTIONS AND THEIR INTEGRATION

As we shall see below, not every elementary function by far has an integral expressed in elementary functions. For this reason, it is very important to separate out those classes of functions whose integrals are expressed in terms of elementary functions. The simplest of these classes is the class of rational functions.

Every rational function may be represented in the form of a rational fraction, that is to say, as a ratio of two polynomials:

$$\frac{Q(x)}{f(x)}=\frac{B_0x^m+B_1x^{m-1}+\ldots+B_m}{A_0x^n+A_1x^{n-1}+\ldots+A_n}$$

Without restricting the generality of our reasoning, we shall assume that these polynomials do not have common roots.

If the degree of the numerator is lower than that of the denominator, then the fraction is called *proper*, otherwise the fraction is called *improper*.

If the fraction is an improper one, then by dividing the numerator by the denominator (by the rule for division of polynomials), it is possible to represent the fraction as the sum of a polynomial and a proper fraction:

$$\frac{Q(x)}{f(x)}=M(x)+\frac{F(x)}{f(x)}$$

Here $M(x)$ is a polynomial, and $\frac{F(x)}{f(x)}$ is a proper fraction.

Example 1. Given an improper rational fraction

$$\frac{x^4-3}{x^2+2x+1}$$

Dividing the numerator by the denominator (by the rule for division of polynomials), we get

$$\frac{x^4-3}{x^2+2x+1}=x^2-2x+3-\frac{4x-6}{x^2+2x+1}$$

Since integration of polynomials does not present any difficulties, the basic barrier when integrating rational fractions is the integration of *proper* rational fractions.

Definition. Proper rational fractions of the form:

I. $\frac{A}{x-a}$,

II. $\frac{A}{(x-a)^k}$ (k a positive integer $\geqslant 2$),

III. $\frac{Ax+B}{x^2+px+q}$ $\left(\text{the roots of the denominator are complex, that is, } \frac{p^2}{4}-q<0\right)$,

IV. $\frac{Ax+B}{(x^2+px+q)^k}$ (k a positive integer $\geqslant 2$; the roots of the denominator are complex) are called *partial fractions of types* I, II, III, *and* IV.

It will be proved below (see Sec. 10.8) that every rational fraction may be represented as a sum of partial fractions. We shall therefore first consider integrals of partial fractions.

The integration of partial fractions of types I, II and III does not present any particular difficulties so we shall perform their integration without any remarks:

I. $\int \frac{A}{x-a}\,dx = A\ln|x-a| + C.$

II. $\int \frac{A}{(x-a)^k}\,dx = A\int (x-a)^{-k}\,dx = A\frac{(x-a)^{-k+1}}{-k+1} + C$

$$= \frac{A}{(1-k)(x-a)^{k-1}} + C.$$

III. $\int \frac{Ax+B}{x^2+px+q}\,dx = \int \frac{\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)}{x^2+px+q}\,dx$

$$= \frac{A}{2}\int \frac{2x+p}{x^2+px+q}\,dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{x^2+px+q}$$

$$= \frac{A}{2}\ln|x^2+px+q| + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)}$$

$= \frac{A}{2}\ln|x^2+px+q| + \frac{2B-Ap}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}} + C$ (see Sec. 10.5).

The integration of partial fractions of type IV requires more involved computations. Suppose we have an integral of this type:

IV. $\int \frac{Ax+B}{(x^2+px+q)^k}\,dx.$

Perform the transformations:

$$\int \frac{Ax+B}{(x^2+px+q)^k}\,dx = \int \frac{\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)}{(x^2+px+q)^k}\,dx$$

$$= \frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}\,dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{(x^2+px+q)^k}$$

The first integral is taken via the substitution, $x^2+px+q=t$, $(2x+p)\,dx=dt$:

$$\int \frac{2x+p}{(x^2+px+q)^k}\,dx = \int \frac{dt}{t^k} = \int t^{-k}\,dt = \frac{t^{-k+1}}{1-k} + C$$

$$= \frac{1}{(1-k)(x^2+px+q)^{k-1}} + C$$

We write the second integral (let us denote it by I_k) in the form

$$I_k=\int\frac{dx}{(x^2+px+q)^k}=\int\frac{dx}{\left[\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right]^k}=\int\frac{dt}{(t^2+m^2)^k}$$

setting

$$x+\frac{p}{2}=t,\quad dx=dt,\quad q-\frac{p^2}{4}=m^2$$

$\Big($it is assumed that the roots of the denominator are complex, and hence, $q-\frac{p^2}{4}>0\Big)$. We then do as follows:

$$I_k=\int\frac{dt}{(t^2+m^2)^k}=\frac{1}{m^2}\int\frac{(t^2+m^2)-t^2}{(t^2+m^2)^k}dt$$

$$=\frac{1}{m^2}\int\frac{dt}{(t^2+m^2)^{k-1}}-\frac{1}{m^2}\int\frac{t^2}{(t^2+m^2)^k}dt \qquad (1)$$

We transform the last integral:

$$\int\frac{t^2\,dt}{(t^2+m^2)^k}=\int\frac{t\cdot t\,dt}{(t^2+m^2)^k}$$

$$=\frac{1}{2}\int t\,\frac{d(t^2+m^2)}{(t^2+m^2)^k}=-\frac{1}{2(k-1)}\int t\,d\left(\frac{1}{(t^2+m^2)^{k-1}}\right)$$

Integrating by parts we get

$$\int\frac{t^2\,dt}{(t^2+m^2)^k}=-\frac{1}{2(k-1)}\left[t\,\frac{1}{(t^2+m^2)^{k-1}}-\int\frac{dt}{(t^2+m^2)^{k-1}}\right]$$

Putting this expression into (1), we have

$$I_k=\int\frac{dt}{(t^2+m^2)^k}=\frac{1}{m^2}\int\frac{dt}{(t^2+m^2)^{k-1}}$$

$$+\frac{1}{m^2}\frac{1}{2(k-1)}\left[\frac{t}{(t^2+m^2)^{k-1}}-\int\frac{dt}{(t^2+m^2)^{k-1}}\right]$$

$$=\frac{t}{2m^2(k-1)(t^2+m^2)^{k-1}}-\frac{2k-3}{2m^2(k-1)}\int\frac{dt}{(t^2+m^2)^{k-1}}$$

On the right side is an integral of the same type as I_k, but the exponent of the denominator of the integrand is less by unity ($k-1$); we have thus expressed I_k in terms of I_{k-1}.

Continuing in the same manner we will arrive at the familiar integral

$$I_1=\int\frac{dt}{t^2+m^2}=\frac{1}{m}\arctan\frac{t}{m}+C$$

Then substituting everywhere in place of t and m their values, we get the expression of integral IV in terms of x and the given numbers A, B, p, q.

Example 2.

$$\int \frac{x-1}{(x^2+2x+3)^2}\,dx = \int \frac{\frac{1}{2}(2x+2)+(-1-1)}{(x^2+2x+3)^2}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{(x^2+2x+3)^2}\,dx - 2\int \frac{dx}{(x^2+2x+3)^2}$$

$$= -\frac{1}{2}\,\frac{1}{(x^2+2x+3)} - 2\int \frac{dx}{(x^2+2x+3)^2}$$

We apply the substitution $x+1=t$ to the last integral:

$$\int \frac{dx}{(x^2+2x+3)^2} = \int \frac{dx}{[(x+1)^2+2]^2} = \int \frac{dt}{(t^2+2)^2} = \frac{1}{2}\int \frac{(t^2+2)-t^2}{(t^2+2)^2}\,dt$$

$$= \frac{1}{2}\int \frac{dt}{t^2+2} - \frac{1}{2}\int \frac{t^2}{(t^2+2)^2}\,dt$$

$$= \frac{1}{2}\,\frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}} - \frac{1}{2}\int \frac{t^2\,dt}{(t^2+2)^2}$$

Let us consider the last integral:

$$\int \frac{t^2\,dt}{(t^2+2)^2} = \frac{1}{2}\int \frac{t\,d(t^2+2)}{(t^2+2)^2} = -\frac{1}{2}\int t\,d\left(\frac{1}{t^2+2}\right)$$

$$= -\frac{1}{2}\,\frac{t}{t^2+2} + \frac{1}{2}\int \frac{dt}{t^2+2}$$

$$= -\frac{t}{2(t^2+2)} + \frac{1}{2\sqrt{2}}\arctan\frac{t}{\sqrt{2}}$$

(we do not yet write the arbitrary constant but will take it into account in the final result).

Consequently,

$$\int \frac{dx}{(x^2+2x+3)^2} = \frac{1}{2\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}$$

$$-\frac{1}{2}\left[-\frac{x+1}{2(x^2+2x+3)} + \frac{1}{2\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}\right]$$

Finally we get

$$\int \frac{x-1}{(x^2+2x+3)^2}\,dx = -\frac{x+2}{2(x^2+2x+3)} - \frac{\sqrt{2}}{4}\arctan\frac{x+1}{\sqrt{2}} + C$$

10.8 DECOMPOSITION OF A RATIONAL FRACTION INTO PARTIAL FRACTIONS

We shall now show that every proper rational fraction may be decomposed into a sum of partial fractions.

Suppose we have a proper rational fraction

$$\frac{F(x)}{f(x)}$$

We shall assume that the coefficients of the polynomials are real numbers and that the given fraction is in lowest terms (this means that the numerator and denominator do not have common roots).

Theorem 1. *Let $x=a$ be a root of multiplicity k of the denominator; that is $f(x)=(x-a)^k f_1(x)$, where $f_1(a)\neq 0$* (see Sec. 7.6). *Then the given proper fraction $\frac{F(x)}{f(x)}$ may be represented in the form of a sum of two other proper fractions as follows:*

$$\frac{F(x)}{f(x)}=\frac{A}{(x-a)^k}+\frac{F_1(x)}{(x-a)^{k-1}f_1(x)} \tag{1}$$

where A is a nonzero constant, and $F_1(x)$ is a polynomial of degree less than the degree of the denominator $(x-a)^{k-1}f_1(x)$.

Proof. Let us write the identity

$$\frac{F(x)}{f(x)}=\frac{A}{(x-a)^k}+\frac{F(x)-Af_1(x)}{(x-a)^k f_1(x)} \tag{2}$$

(which is true for every A) and let us define the constant A so that the polynomial $F(x)-Af_1(x)$ can be divided by $x-a$. To do this, by the remainder theorem, it is necessary and sufficient that the following equation hold:

$$F(a)-Af_1(a)=0$$

Since $f_1(a)\neq 0$, $F(a)\neq 0$, A is uniquely defined by

$$A=\frac{F(a)}{f_1(a)}$$

For such an A we shall have

$$F(x)-Af_1(x)=(x-a)F_1(x)$$

where $F_1(x)$ is a polynomial of degree less than that of the polynomial $(x-a)^{k-1}f_1(x)$. Cancelling $(x-a)$ from the fraction in formula (2), we get (1).

Corollary. Similar reasoning may be applied to the proper rational fraction

$$\frac{F_1(x)}{(x-a)^{k-1}f_1(x)}$$

in equation (1). Thus, if the denominator has a root $x=a$ of multiplicity k, we can write

$$\frac{F(x)}{f(x)}=\frac{A}{(x-a)^k}+\frac{A_1}{(x-a)^{k-1}}+\ldots+\frac{A_{k-1}}{x-a}+\frac{F_k(x)}{f_1(x)}$$

where $\frac{F_k(x)}{f_1(x)}$ is a proper fraction in lowest terms. To it we can apply the theorem that has just been proved, provided $f_1(x)$ has other real roots.

Let us now consider the case of complex roots of the denominator. Recall that the complex roots of a polynomial with real coefficients are always conjugate in pairs (see Sec. 7.8).

When factoring a polynomial into real factors, to each pair of complex roots of the polynomial there corresponds an expression of the form x^2+px+q. But if the complex roots are of multiplicity μ, they correspond to the expression $(x^2+px+q)^{\mu}$.

Theorem 2. *If* $f(x)=(x^2+px+q)^{\mu}\varphi_1(x)$, *where the polynomial* $\varphi_1(x)$ *is not divisible by* x^2+px+q, *then the proper rational fraction* $\frac{F(x)}{f(x)}$ *may be represented as a sum of two other proper fractions in the following manner:*

$$\frac{F(x)}{f(x)}=\frac{Mx+N}{(x^2+px+q)^{\mu}}+\frac{\Phi_1(x)}{(x^2+px+q)^{\mu-1}\varphi_1(x)} \tag{3}$$

where $\Phi_1(x)$ *is a polynomial of degree less than that of the polynomial* $(x^2+px+q)^{\mu-1}\varphi_1(x)$.

Proof. Let us write the identity

$$\frac{F(x)}{f(x)}=\frac{F(x)}{(x^2+px+q)^{\mu}\varphi_1(x)}=\frac{Mx+N}{(x^2+px+q)^{\mu}}+\frac{F(x)-(Mx+N)\varphi_1(x)}{(x^2+px+q)^{\mu}\varphi_1(x)} \tag{4}$$

which is true for all M and N, and let us define M and N so that the polynomial $F(x)-(Mx+N)\varphi_1(x)$ is divisible by x^2+px+q. To do this, it is necessary and sufficient that the equation

$$F(x)-(Mx+N)\varphi_1(x)=0$$

have the same roots $\alpha\pm i\beta$ as the polynomial x^2+px+q. Thus,

$$F(\alpha+i\beta)-[M(\alpha+i\beta)+N]\varphi_1(\alpha+i\beta)=0$$

or

$$M(\alpha+i\beta)+N=\frac{F(\alpha+i\beta)}{\varphi_1(\alpha+i\beta)}$$

But $\frac{F(\alpha+i\beta)}{\varphi_1(\alpha+i\beta)}$ is a definite complex number which may be written in the form $K+iL$, where K and L are certain real numbers. Thus,

$$M(\alpha+i\beta)+N=K+iL$$

whence

$$M\alpha+N=K,\quad M\beta=L$$

or

$$M=\frac{L}{\beta},\quad N=\frac{K\beta-L\alpha}{\beta}$$

With these values of the coefficients M and N the polynomial $F(x)-(Mx+N)\varphi_1(x)$ has the number $\alpha+i\beta$ for a root, and, hence, also the conjugate number $\alpha-i\beta$. But then the polynomial can be divided, without remainder, by the differences $x-(\alpha+i\beta)$ and $x-(\alpha-i\beta)$, and, therefore, by their product, which is x^2+px+q. Denoting the quotient of this division by $\Phi_1(x)$, we get

$$F(x)-(Mx+N)\varphi_1(x)=(x^2+px+q)\Phi_1(x)$$

Cancelling x^2+px+q from the last fraction in (4), we get (3), and it is clear that the degree of $\Phi_1(x)$ is less than that of the denominator, which is what we set out to prove.

Now applying to the proper fraction $\frac{F(x)}{f(x)}$ the results of Theorems 1 and 2, we can obtain, successively, all the partial fractions corresponding to all the roots of the denominator $f(x)$. Thus, from the foregoing the result follows that

If

$$f(x)=(x-a)^{\alpha}\ldots(x-b)^{\beta}(x^2+px+q)^{\mu}\ldots(x^2+lx+s)^{\nu},$$

then the fraction $\frac{F(x)}{f(x)}$ *can be represented as follows:*

$$\left.\begin{aligned}\frac{F(x)}{f(x)}=&\frac{A}{(x-a)^{\alpha}}+\frac{A_1}{(x-a)^{\alpha-1}}+\ldots+\frac{A_{\alpha-1}}{x-a}\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &+\frac{B}{(x-b)^{\beta}}+\frac{B_1}{(x-b)^{\beta-1}}+\ldots+\frac{B_{\beta-1}}{x-b}\\ &+\frac{Mx+N}{(x^2+px+q)^{\mu}}+\frac{M_1x+N_1}{(x^2+px+q)^{\mu-1}}+\ldots+\frac{M_{\mu-1}x+N_{\mu-1}}{x^2+px+q}\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &+\frac{Px+Q}{(x^2+lx+s)^{\nu}}+\frac{P_1x+Q_1}{(x^2+lx+s)^{\nu-1}}+\ldots+\frac{P_{\nu-1}x+Q_{\nu-1}}{x^2+lx+s}\end{aligned}\right\}\quad(5)$$

The coefficients A, A_1, ..., B, B_1, ... may be determined by the following reasoning. This equality is an **identity**; and for this reason, by reducing the fractions to a common denominator we get identical polynomials in the numerators on the right and left. Equating the coefficients of the same degrees of x, we get a system of equations for determining the unknown coefficients A, A_1, ..., B, B_1, ... This method of finding coefficients is called the *method of undetermined coefficients.*

Besides, to determine the coefficients we can take advantage of the following: since the polynomials obtained on the right and left sides of the equation must be identically equal after reducing to a common denominator, their values are equal for all particular values of x. Assigning particular values to x, we get equations for determining the coefficients.

We thus see that every proper rational fraction may be represented in the form of a sum of partial rational fractions.

Example. Let it be required to decompose the fraction $\frac{x^2+2}{(x+1)^3(x-2)}$ into partial fractions. From (5) we have

$$\frac{x^2+2}{(x+1)^3(x-2)}=\frac{A}{(x+1)^3}+\frac{A_1}{(x+1)^2}+\frac{A_2}{x+1}+\frac{B}{x-2}$$

Reducing to a common denominator and equating the numerators, we have

$$x^2+2=A(x-2)+A_1(x+1)(x-2)+A_2(x+1)^2(x-2)+B(x+1)^3 \qquad (6)$$

or

$$x^2+2=(A_2+B)x^3+(A_1+3B)x^2$$
$$+(A-A_1-3A_2+3B)x+(-2A-2A_1-2A_2+B)$$

Equating the coefficients of x^3, x^2, x^1, x^0 (absolute term), we get a system of equations for determining the coefficients:

$$0=A_2+B$$
$$1=A_1+3B$$
$$0=A-A_1-3A_2+3B$$
$$2=-2A-2A_1-2A_2+B$$

Solving this system we find

$$A=-1; \quad A_1=\frac{1}{3}; \quad A_2=-\frac{2}{9}; \quad B=\frac{2}{9}$$

It might also be possible to determine some of the coefficients of the equations that result for some particular values of x from (6), which is an identity in x. Thus, setting $x=-1$ we have $3=-3A$ or $A=-1$; setting $x=2$, we have $6=27B$, $B=\frac{2}{9}$.

If to these two equations we add two equations that result from equating the coefficients of the same powers of x, we get four equations for determining the four unknown coefficients. As a result, we have the decomposition

$$\frac{x^2+2}{(x+1)^3(x-2)}=-\frac{1}{(x+1)^3}+\frac{1}{3(x+1)^2}-\frac{2}{9(x+1)}+\frac{2}{9(x-2)}$$

10.9 INTEGRATION OF RATIONAL FRACTIONS

Let it be required to evaluate the integral of a rational fraction $\frac{Q(x)}{f(x)}$; that is, the integral

$$\int \frac{Q(x)}{f(x)}\,dx$$

If the given fraction is **improper**, we represent it as the sum of a polynomial $M(x)$ and the **proper** rational fraction $\frac{F(x)}{f(x)}$ (see Sec. 10.7). This latter we represent, applying formula (5), Sec. 10.8, as a sum of **partial** fractions. Thus, the integration of a rational fraction reduces to the integration of a polynomial and several **partial** fractions.

From the results of Sec. 10.8 it follows that the form of partial fractions is determined by the roots of the denominator $f(x)$. The following cases are possible.

Case I. *The roots of the denominator are real and distinct, that is*

$$f(x)=(x-a)(x-b)\ldots(x-d)$$

Here, the fraction $\frac{F(x)}{f(x)}$ is decomposable into partial fractions of type I:

$$\frac{F(x)}{f(x)}=\frac{A}{x-a}+\frac{B}{x-b}+\ldots+\frac{D}{x-d}$$

and then

$$\int\frac{F(x)}{f(x)}dx=\int\frac{A}{x-a}dx+\int\frac{B}{x-b}dx+\ldots+\int\frac{D}{x-d}dx$$
$$=A\ln|x-a|+B\ln|x-b|+\ldots+D\ln|x-d|+C$$

Case II. *The roots of the denominator are real, and some of them are multiple:*

$$f(x)=(x-a)^{\alpha}(x-b)^{\beta}\ldots(x-d)^{\delta}$$

In this case the fraction $\frac{F(x)}{f(x)}$ is decomposable into partial fractions of types I and II.

Example 1 (see example in Sec. 10.8).

$$\int\frac{x^2+2}{(x+1)^3(x-2)}dx=-\int\frac{dx}{(x+1)^3}+\frac{1}{3}\int\frac{dx}{(x+1)^2}-\frac{2}{9}\int\frac{dx}{x+1}$$
$$+\frac{2}{9}\int\frac{dx}{x-2}=\frac{1}{2}\frac{1}{(x+1)^2}-\frac{1}{3(x+1)}-\frac{2}{9}\ln|x+1|+\frac{2}{9}\ln|x-2|+C$$
$$=-\frac{2x-1}{6(x+1)^2}+\frac{2}{9}\ln\left|\frac{x-2}{x+1}\right|+C$$

Case III. *Among the roots of the denominator are complex non repeated (that is, distinct) roots:*

$$f(x)=(x^2+px+q)\ldots(x^2+lx+s)(x-a)^{\alpha}\ldots(x-d)^{\delta}$$

In this case the fraction $\frac{F(x)}{f(x)}$ is decomposable into partial fractions of types I, II, and III.

Example 2. Evaluate the integral

$$\int\frac{x\,dx}{(x^2+1)(x-1)}$$

Decompose the fraction under the integral sign into partial fractions [see (5), Sec. 10.8]

$$\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}$$

Consequently,

$$x=(Ax+B)(x-1)+C(x^2+1)$$

Setting $x=1$, we get $1=2C$, $C=\frac{1}{2}$; setting $x=0$, we get $0=-B+C$, $B=\frac{1}{2}$.

Equating the coefficients of x^2, we get $0=A+C$, whence $A=-\frac{1}{2}$. Thus,

$$\int\frac{x\,dx}{(x^2+1)(x-1)}=-\frac{1}{2}\int\frac{x-1}{x^2+1}dx+\frac{1}{2}\int\frac{dx}{x-1}$$

$$=-\frac{1}{2}\int\frac{x\,dx}{x^2+1}+\frac{1}{2}\int\frac{dx}{x^2+1}+\frac{1}{2}\int\frac{dx}{x-1}$$

$$=-\frac{1}{4}\ln|x^2+1|+\frac{1}{2}\arctan x+\frac{1}{2}\ln|x-1|+C$$

Case IV. *The roots of the denominator include complex multiple roots*

$$f(x)=(x^2+px+q)^\mu\ldots(x^2+lx+s)^\nu(x-a)^\alpha\ldots(x-d)^\delta$$

In this case, decomposition of the fraction $\frac{F(x)}{f(x)}$ will also contain partial fractions of type IV.

Example 3. It is required to evaluate the integral

$$\int\frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}dx$$

Solution. Decompose the fraction into partial fractions:

$$\frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}=\frac{Ax+B}{(x^2+2x+3)^2}+\frac{Cx+D}{(x^2+2x+3)}+\frac{E}{x+1}$$

whence

$$x^4+4x^3+11x^2+12x+8$$
$$=(Ax+B)(x+1)+(Cx+D)(x^2+2x+3)(x+1)+E(x^2+2x+3)^2$$

Combining the above-indicated methods of determining coefficients, we find

$$A=1,\quad B=-1,\quad C=0,\quad D=0,\quad E=1$$

Thus, we get

$$\int\frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}dx=\int\frac{x-1}{(x^2+2x+3)^2}dx+\int\frac{dx}{x+1}$$

$$=-\frac{x+2}{2(x^2+2x+3)}-\frac{\sqrt{2}}{4}\arctan\frac{x+1}{\sqrt{2}}+\ln|x+1|+C$$

The first integral on the right was considered in Example 2, Sec. 10.7. The second integral is taken directly.

From the foregoing it follows that the integral of any rational function may be expressed in terms of elementary functions in closed form, namely, in terms of:

(1) logarithms in the case of partial fractions of type I;

(2) rational functions in the case of partial fractions of type II;

(3) logarithms and arc tangents in the case of partial fractions of type III;

(4) rational functions and arc tangents in the case of partial fractions of type IV.

10.10 INTEGRALS OF IRRATIONAL FUNCTIONS

It is impossible to express in terms of elementary functions the integral of every irrational function. In this and the following sections we shall consider irrational functions whose integrals are reduced (by means of substitution) to integrals of rational functions and, consequently, are integrated completely.

I. We consider the integral $\int R\left(x, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}\right) dx$ where R is a rational function of its arguments. *

Let k be a common denominator of the fractions $\frac{m}{n}, \ldots, \frac{r}{s}$. We make the substitution

$$x = t^k, \quad dx = kt^{k-1}\,dt$$

Then each **fractional** power of x will be expressed in terms of an **integral** power of t and the integrand will thus be transformed into a **rational function of** t.

Example 1. It is required to compute the integral

$$\int \frac{x^{\frac{1}{2}}\,dx}{x^{\frac{3}{4}}+1}$$

---

* The notation $R\left(x, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}\right)$ indicates that only **rational** operations are performed on the quantities $x, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}$.

This is precisely the way that the following notations are henceforward to be understood: $R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \ldots\right)$, $R(x, \sqrt{ax^2+bx+c})$, $R(\sin x, \cos x)$, etc. For instance, the notation $R(\sin x, \cos x)$ indicates that rational operations are to be performed on $\sin x$ and $\cos x$.

Solution. The common denominator of the fractions $\frac{1}{2}$, $\frac{3}{4}$ is 4; and so we substitute $x=t^4$, $dx=4t^3\,dt$; then

$$\int \frac{x^{\frac{1}{2}}\,dx}{x^{\frac{3}{4}}+1}=4\int \frac{t^2}{t^3+1}t^3\,dt=4\int \frac{t^5}{t^3+1}\,dt=4\int\left(t^2-\frac{t^2}{t^3+1}\right)dt$$

$$=4\int t^2\,dt-4\int\frac{t^2}{t^3+1}\,dt=4\frac{t^3}{3}-\frac{4}{3}\ln|t^3+1|+C$$

$$=\frac{4}{3}\left[x^{\frac{3}{4}}-\ln\left|x^{\frac{3}{4}}+1\right|\right]+C$$

II. Now consider an integral of the form

$$\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right]dx$$

This integral reduces to the integral of a rational function by means of the substitution

$$\frac{ax+b}{cx+d}=t^k$$

where k is the common denominator of the fractions $\frac{m}{n}, \ldots, \frac{r}{s}$.

Example 2. It is required to compute the integral

$$\int\frac{\sqrt{x+4}}{x}\,dx$$

Solution. We make the substitution $x+4=t^2$, $x=t^2-4$, $dx=2t\,dt$; then

$$\int\frac{\sqrt{x+4}}{x}\,dx=2\int\frac{t^2}{t^2-4}\,dt=2\int\left(1+\frac{4}{t^2-4}\right)dt=2\int dt+8\int\frac{dt}{t^2-4}$$

$$=2t+2\ln\left|\frac{t-2}{t+2}\right|+C=2\sqrt{x+4}+2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right|+C$$

10.11 INTEGRALS OF THE FORM $\int R(x, \sqrt{ax^2+bx+c})\,dx$

Let us consider the integral

$$\int R\left(x, \sqrt{ax^2+bx+c}\right)dx \quad (a\neq 0) \tag{1}$$

An integral of this kind reduces to the integral of a rational function of a new variable by means of the following Euler substitutions.

First Euler substitution. If $a>0$, then we put

$$\sqrt{ax^2+bx+c}=\pm\sqrt{a}\,x+t$$

For the sake of definiteness we take the plus sign in front of $\sqrt{a}$. Then

$$ax^2+bx+c=ax^2+2\sqrt{a}\,xt+t^2$$

whence x is determined as a rational function of t:

$$x=\frac{t^2-c}{b-2\sqrt{a}t}$$

(thus, dx will also be expressed rationally in terms of t). Therefore,

$$\sqrt{ax^2+bx+c}=\sqrt{a}\,x+t=\sqrt{a}\,\frac{t^2-c}{b-2t\sqrt{a}}+t$$

Thus $\sqrt{ax^2+bx+c}$ is a rational function of t.

Since $\sqrt{ax^2+bx+c}$, x and dx are expressed rationally in terms of t, the given integral (1) is transformed into an integral of a rational function of t.

Example 1. It is required to compute the integral

$$\int\frac{dx}{\sqrt{x^2+c}}$$

Solution. Since here $a=1>0$, we put $\sqrt{x^2+c}=-x+t$; then

$$x^2+c=x^2-2xt+t^2$$

whence

$$x=\frac{t^2-c}{2t}$$

Consequently,

$$dx=\frac{t^2+c}{2t^2}\,dt$$

$$\sqrt{x^2+c}=-x+t=-\frac{t^2-c}{2t}+t=\frac{t^2+c}{2t}$$

Returning to the original integral, we have

$$\int\frac{dx}{\sqrt{x^2+c}}=\int\frac{\frac{t^2+c}{2t^2}\,dt}{\frac{t^2+c}{2t}}=\int\frac{dt}{t}=\ln|t|+C_1=\ln|x+\sqrt{x^2+c}|+C_1$$

(see formula 14 in the Table of Integrals).

Second Euler substitution. If $c>0$, we put

$$\sqrt{ax^2+bx+c}=xt\pm\sqrt{c}$$

then

$$ax^2+bx+c=x^2t^2+2xt\sqrt{c}+c$$

(For the sake of definiteness we took the plus sign in front of the radical.) Then x is determined as a rational function of t:

$$x=\frac{2\sqrt{c}\,t-b}{a-t^2}.$$

Since dx and $\sqrt{ax^2+bx+c}$ are also expressed rationally in terms of t, by substituting the values of x, $\sqrt{ax^2+bx+c}$ and dx into the integral $\int R(x, \sqrt{ax^2+bx+c})\,dx$, we reduce it to an integral of a rational function of t.

Example 2. It is required to compute the integral

$$\int \frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{1+x+x^2}}\,dx$$

Solution. We set $\sqrt{1+x+x^2}=xt+1$, then

$$1+x+x^2=x^2t^2+2xt+1, \quad x=\frac{2t-1}{1-t^2}, \quad dx=\frac{2t^2-2t+2}{(1-t^2)^2}\,dt$$

$$\sqrt{1+x+x^2}=xt+1=\frac{t^2-t+1}{1-t^2}$$

$$1-\sqrt{1+x+x^2}=\frac{-2t^2+t}{1-t^2}$$

Putting the expressions obtained into the original integral, we find

$$\int \frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{1+x+x^2}}\,dx=\int \frac{(-2t^2+t)^2(1-t^2)^2(1-t^2)(2t^2-2t+2)}{(1-t^2)^2(2t-1)^2(t^2-t+1)(1-t^2)^2}\,dt$$

$$=+2\int \frac{t^2}{1-t^2}\,dt=-2t+\ln\left|\frac{1+t}{1-t}\right|+C$$

$$=-\frac{2(\sqrt{1+x+x^2}-1)}{x}+\ln\left|\frac{x+\sqrt{1+x+x^2}-1}{x-\sqrt{1+x+x^2}+1}\right|+C$$

$$=-\frac{2(\sqrt{1+x+x^2}-1)}{x}+\ln\left|2x+2\sqrt{1+x+x^2}+1\right|+C$$

Third Euler substitution. Let α and β be the real roots of the trinomial ax^2+bx+c. We put

$$\sqrt{ax^2+bx+c}=(x-\alpha)\,t$$

Since $ax^2+bx+c=a(x-\alpha)(x-\beta)$, we have

$$\sqrt{a(x-\alpha)(x-\beta)}=(x-\alpha)\,t$$

$$a(x-\alpha)(x-\beta)=(x-\alpha)^2t^2$$

$$a(x-\beta)=(x-\alpha)\,t^2$$

Whence we find x as a rational function of t:

$$x=\frac{a\beta-\alpha t^2}{a-t^2}$$

Since dx and $\sqrt{ax^2+bx+c}$ also rationally depend upon t, the given integral is transformed into an integral of a rational function of t.

Note 1. The third Euler substitution is applicable not only for $a<0$, but also for $a>0$, provided the polynomial ax^2+bx+c has two real roots.

Example 3. It is required to compute the integral

$$\int \frac{dx}{\sqrt{x^2+3x-4}}$$

Solution. Since $x^2+3x-4=(x+4)(x-1)$, we put

$$\sqrt{(x+4)(x-1)}=(x+4)t$$

then

$$(x+4)(x-1)=(x+4)^2 t^2, \quad x-1=(x+4)t^2$$

$$x=\frac{1+4t^2}{1-t^2}, \quad dx=\frac{10t}{(1-t^2)^2}dt$$

$$\sqrt{(x+4)(x-1)}=\left[\frac{1+4t^2}{1-t^2}+4\right]t=\frac{5t}{1-t^2}$$

Returning to the original integral, we have

$$\int \frac{dx}{\sqrt{x^2+3x-4}}=\int \frac{10t(1-t^2)}{(1-t^2)^2 5t}dt=\int \frac{2}{1-t^2}dt=\ln\left|\frac{1+t}{1-t}\right|+C$$

$$=\ln\left|\frac{1+\sqrt{\frac{x-1}{x+4}}}{1-\sqrt{\frac{x-1}{x+4}}}\right|+C=\ln\left|\frac{\sqrt{x+4}+\sqrt{x-1}}{\sqrt{x+4}-\sqrt{x-1}}\right|+C$$

Note 2. It will be noted that to reduce integral (1) to an integral of a rational function, the first and third Euler substitutions are sufficient. Let us consider the trinomial ax^2+bx+c. If $b^2-4ac>0$, then the roots of the trinomial are real, and, hence, the third Euler substitution is applicable. If $b^2-4ac\leqslant 0$, then in this case

$$ax^2+bx+c=\frac{1}{4a}[(2ax+b)^2+(4ac-b^2)]$$

and therefore the trinomial has the same sign as that of a. For $\sqrt{ax^2+bx+c}$ to be real it is necessary that the trinomial be positive, and we must have $a>0$. In this case, the first substitution is applicable.

10.12 INTEGRATION OF CERTAIN CLASSES OF TRIGONOMETRIC FUNCTIONS

Up to now our sole concern has been a systematic study of the integrals of algebraic functions (rational and irrational). In this section we shall consider integrals of certain classes of nonalgebraic functions, primarily trigonometric. Let us consider an integral of the form

$$\int R(\sin x,\ \cos x)\,dx \tag{1}$$

We shall show that this integral, by the substitution

$$\tan\frac{x}{2}=t \tag{2}$$

always reduces to an integral of a rational function. Let us express $\sin x$ and $\cos x$ in terms of $\tan\frac{x}{2}$, and hence, in terms of t:

$$\sin x=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2t}{1+t^2}$$

$$\cos x=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{1}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

Furthermore,

$$x=2\arctan t,\quad dx=\frac{2dt}{1+t^2}$$

In this way, $\sin x$, $\cos x$ and dx are expressed rationally in terms of t. Since a rational function of rational functions is a rational function, by substituting the expressions obtained into the integral (1) we get an integral of a rational function:

$$\int R(\sin x,\ \cos x)\,dx=\int R\left(\frac{2t}{1+t^2},\ \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}$$

Example 1. Consider the integral

$$\int\frac{dx}{\sin x}$$

On the basis of the foregoing formulas we have

$$\int\frac{dx}{\sin x}=\int\frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}}=\int\frac{dt}{t}=\ln|t|+C=\ln\left|\tan\frac{x}{2}\right|+C$$

This substitution enables us to integrate any function of the form $R(\cos x,\ \sin x)$. For this reason it is sometimes called a "universal trigonometric substitution". However, in practice it frequently leads to extremely complex rational functions. It is therefore convenient to know some other substitutions (in addition to the "universal" one) that sometimes lead more quickly to the desired end.

(1) If an integral is of the form $\int R(\sin x)\cos x\,dx$, the substitution $\sin x = t$, $\cos x\,dx = dt$ reduces this integral to the form $\int R(t)\,dt$.

(2) If the integral has the form $\int R(\cos x)\sin x\,dx$, it is reduced to an integral of a rational function by the substitution $\cos x = t$, $\sin x\,dx = -dt$.

(3) If the integrand is dependent only on $\tan x$, then the substitution $\tan x = t$, $x = \arctan t$, $dx = \frac{dt}{1+t^2}$ reduces this integral to an integral of a rational function:

$$\int R(\tan x)\,dx = \int R(t)\frac{dt}{1+t^2}$$

(4) If the integrand has the form $R(\sin x, \cos x)$, but $\sin x$ and $\cos x$ are involved only in **even** powers, then the same substitution is applied:

$$\tan x = t \tag{2'}$$

because $\sin^2 x$ and $\cos^2 x$ can be expressed rationally in terms of $\tan x$:

$$\cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2}$$

$$\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x} = \frac{t^2}{1+t^2}$$

$$dx = \frac{dt}{1+t^2}$$

After the substitution we obtain an integral of a rational function.

Example 2. Compute the integral $\int \frac{\sin^3 x}{2+\cos x}dx$.

Solution. This integral is readily reduced to the form $\int R(\cos x)\sin x\,dx$. Indeed,

$$\int \frac{\sin^3 x}{2+\cos x}dx = \int \frac{\sin^2 x \sin x\,dx}{2+\cos x} = \int \frac{1-\cos^2 x}{2+\cos x}\sin x\,dx$$

We make the substitution $\cos x = z$. Then $\sin x\,dx = -dz$:

$$\int \frac{\sin^3 x}{2+\cos x}dx = \int \frac{1-z^2}{2+z}(-dz) = \int \frac{z^2-1}{z+2}dz = \int \left(z-2+\frac{3}{z+2}\right)dz$$

$$= \frac{z^2}{2} - 2z + 3\ln(z+2) + C = \frac{\cos^2 x}{2} - 2\cos x + 3\ln(\cos x + 2) + C$$

Example 3. Compute $\int \frac{dx}{2-\sin^2 x}$.

Make the substitution $\tan x = t$:

$$\int \frac{dx}{2-\sin^2 x} = \int \frac{dt}{\left(2-\frac{t^2}{1+t^2}\right)(1+t^2)} = \int \frac{dt}{2+t^2} = \frac{1}{\sqrt{2}} \arctan \frac{t}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

(5) Now let us consider one more integral of the form $\int R(\sin x, \cos x)\,dx$, namely an integral with integrand $\sin^m x \cos^n x\,dx$ (where m and n are integers). Here we consider three cases.

(a) $\int \sin^m x \cos^n x\,dx$, where m and n are such that at least one of them is **odd**. For definiteness let us assume that n is odd. Put $n = 2p+1$ and transform the integral:

$$\int \sin^m x \cos^{2p+1} x\,dx = \int \sin^m x \cos^{2p} x \cos x\,dx$$

$$= \int \sin^m x (1-\sin^2 x)^p \cos x\,dx$$

Change the variable:

$$\sin x = t, \quad \cos x\,dx = dt$$

Putting the new variable into the given integral, we get

$$\int \sin^m x \cos^n x\,dx = \int t^m (1-t^2)^p\,dt$$

which is an integral of a rational function of t.

Example 4.

$$\int \frac{\cos^3 x}{\sin^4 x}\,dx = \int \frac{\cos^2 x \cos x\,dx}{\sin^4 x} = \int \frac{(1-\sin^2 x)\cos x\,dx}{\sin^4 x}$$

Denoting $\sin x = t$, $\cos x\,dx = dt$, we get

$$\int \frac{\cos^3 x}{\sin^4 x}\,dx = \int \frac{(1-t^2)\,dt}{t^4} = \int \frac{dt}{t^4} - \int \frac{dt}{t^2} = -\frac{1}{3t^3} + \frac{1}{t} + C$$

$$= -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + C$$

(b) $\int \sin^m x \cos^n x\,dx$, where m and n are nonnegative and even numbers.

Put $m = 2p$, $n = 2q$. Write the familiar trigonometric formulas:

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x, \quad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x \qquad (3)$$

Putting them into the integral we get

$$\int \sin^{2p} x \cos^{2q} x\,dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^p \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^q dx$$

Powering and opening brackets, we get terms containing $\cos 2x$ to odd and even powers. The terms with odd powers are integrated as indicated in Case (a). We again reduce the even exponents by formulas (3). Continuing in this manner we arrive at terms of the form $\int \cos kx\,dx$, which can easily be integrated.

Example 5.

$$\int \sin^4 x\,dx = \frac{1}{2^2}\int (1-\cos 2x)^2\,dx = \frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)\,dx$$
$$=\frac{1}{4}\left[x-\sin 2x+\frac{1}{2}\int (1+\cos 4x)\,dx\right]=\frac{1}{4}\left[\frac{3}{2}x-\sin 2x+\frac{\sin 4x}{8}\right]+C$$

(c) If both exponents are even, and at least one of them is negative, then the preceding technique does not give the desired result. Here, one should make the substitution $\tan x = t$ (or $\cot x = t$).

Example 6.

$$\int \frac{\sin^2 x\,dx}{\cos^6 x}=\int \frac{\sin^2 x\,(\sin^2 x+\cos^2 x)^2}{\cos^6 x}\,dx=\int \tan^2 x\,(1+\tan^2 x)^2\,dx$$

Put $\tan x = t$; then $x=\arctan t$, $dx=\frac{dt}{1+t^2}$ and we get

$$\int \frac{\sin^2 x}{\cos^6 x}\,dx=\int t^2(1+t^2)^2\frac{dt}{1+t^2}=\int t^2(1+t^2)\,dt=\frac{t^3}{3}+\frac{t^5}{5}+C$$
$$=\frac{\tan^3 x}{3}+\frac{\tan^5 x}{5}+C$$

(6) In conclusion let us consider integrals of the form

$$\int \cos mx\cos nx\,dx,\qquad \int \sin mx\cos nx\,dx,\qquad \int \sin mx\sin nx\,dx$$

They are taken by means of the following formulas* $(m\neq n)$:

$$\cos mx\cos nx=\frac{1}{2}\left[\cos(m+n)x+\cos(m-n)x\right]$$
$$\sin mx\cos nx=\frac{1}{2}\left[\sin(m+n)x+\sin(m-n)x\right]$$
$$\sin mx\sin nx=\frac{1}{2}\left[-\cos(m+n)x+\cos(m-n)x\right]$$

* These formulas are easily derived as follows:

$$\cos(m+n)x=\cos mx\cos nx-\sin mx\sin nx$$
$$\cos(m-n)x=\cos mx\cos nx+\sin mx\sin nx$$

Combining these equations termwise and dividing them in half, we get the first of the three formulas. Subtracting termwise and dividing in half, we get the third formula. The second formula is similarly derived if we write analogous equations for $\sin(m+n)x$ and $\sin(m-n)x$ and then combine them termwise.

Substituting and integrating, we get

$$\int \cos mx \cos nx\, dx = \frac{1}{2}\int [\cos(m+n)x + \cos(m-n)x]\, dx$$
$$= \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C$$

The other two integrals are evaluated similarly.

Example 7.

$$\int \sin 5x \sin 3x\, dx = \frac{1}{2}\int [-\cos 8x + \cos 2x]\, dx = -\frac{\sin 8x}{16} + \frac{\sin 2x}{4} + C$$

10.13 INTEGRATION OF CERTAIN IRRATIONAL FUNCTIONS BY MEANS OF TRIGONOMETRIC SUBSTITUTIONS

Let us return to the integral considered in Sec. 10.11:

$$\int R(x, \sqrt{ax^2+bx+c})\, dx \tag{1}$$

where $a \neq 0$ and $c - \frac{b^2}{4a} \neq 0$ (in the case $a=0$ the integral has form II, Sec. 10.10; for $c - \frac{b^2}{4a} = 0$, the expression $ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2$, and we have to do with a rational function, if $a > 0$; for $a < 0$ the function $\sqrt{ax^2+bx+c}$ is not defined for any value of x). Here we shall give a method of transforming this integral into one of the form

$$\int \overline{R}(\sin z, \cos z)\, dz \tag{2}$$

which was considered in the preceding section.

Transform the trinomial under the radical sign:

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

Change the variable, putting

$$x + \frac{b}{2a} = t, \quad dx = dt$$

Then

$$\sqrt{ax^2+bx+c} = \sqrt{at^2 + \left(c - \frac{b^2}{4a}\right)}$$

Let us consider all possible cases.

1. Let $a > 0$, $c - \frac{b^2}{4a} > 0$. We introduce the designations: $a = m^2$, $c - \frac{b^2}{4a} = n^2$. In this case we have

$$\sqrt{ax^2+bx+c} = \sqrt{m^2t^2+n^2}$$

2. Let $a > 0$, $c - \frac{b^2}{4a} < 0$. Then

$$a = m^2, \quad c - \frac{b^2}{4a} = -n^2$$

Thus,

$$\sqrt{ax^2 + bx + c} = \sqrt{m^2t^2 - n^2}$$

3. Let $a < 0$, $c - \frac{b^2}{4a} > 0$. Then

$$a = -m^2, \quad c - \frac{b^2}{4a} = n^2$$

Hence,

$$\sqrt{ax^2 + bx + c} = \sqrt{n^2 - m^2t^2}$$

4. Let $a < 0$, $c - \frac{b^2}{4a} < 0$. In this case $\sqrt{ax^2 + bx + c}$ is a complex number for every value of x.

In this way, integral (1) is reduced to one of the following types of integrals:

$$\text{I.} \quad \int R\left(t, \sqrt{m^2t^2 + n^2}\right) dt \tag{3a}$$

$$\text{II.} \quad \int R\left(t, \sqrt{m^2t^2 - n^2}\right) dt \tag{3b}$$

$$\text{III.} \quad \int R\left(t, \sqrt{n^2 - m^2t^2}\right) dt \tag{3c}$$

Obviously, integral (3a) is reduced to an integral of the form (2) by the substitution

$$t = \frac{n}{m} \tan z$$

Integral (3b) is reduced to the form (2) by the substitution

$$t = \frac{n}{m} \sec z$$

Integral (3c) is reduced to (2) by the substitution

$$t = \frac{n}{m} \sin t$$

Example. Compute the integral

$$\int \frac{dx}{\sqrt{(a^2 - x^2)^3}}$$

Solution. This is an integral of type III. Make the substitution $x = a \sin z$, then

$$dx = a\cos z\,dz$$

$$\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \int \frac{a\cos z\,dz}{\sqrt{(a^2-a^2\sin^2 z)^3}} = \int \frac{a\cos z\,dz}{a^3\cos^3 z}{}^{*}$$

$$= \frac{1}{a^2}\int \frac{dz}{\cos^2 z} = \frac{1}{a^2}\tan z + C = \frac{1}{a^2}\frac{\sin z}{\cos z} + C = \frac{1}{a^2}\frac{\sin z}{\sqrt{1-\sin^2 z}} + C$$

$$= \frac{1}{a^2}\frac{x}{\sqrt{a^2-x^2}} + C$$

10.14 ON FUNCTIONS WHOSE INTEGRALS CANNOT BE EXPRESSED IN TERMS OF ELEMENTARY FUNCTIONS

In Sec. 10.1 we pointed out (without proof) that any function $f(x)$ continuous on an interval (a, b) has an antiderivative on that interval; in other words, there exists a function $F(x)$ such that $F'(x) = f(x)$. However, **not every antiderivative,** even when it exists, **is expressible, in closed form, in terms of elementary functions.**

Such are the antiderivatives expressed by the integrals $\int e^{-x^2}dx$, $\int \frac{\sin x}{x}dx$, $\int \frac{\cos x}{x}dx$, $\int \sqrt{1-k^2\sin^2 x}\,dx$, $\int \frac{dx}{\ln x}$ and many others.

In all such cases, the antiderivative is obviously some new function which does not reduce to a combination of a finite number of elementary functions.

For example, that one of the antiderivatives

$$\frac{2}{\sqrt{\pi}}\int e^{-x^2}dx + C$$

which vanishes for $x=0$ is called the *Laplace function* and is denoted by $\Phi(x)$. Thus,

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int e^{-x^2}dx + C_1 \text{ if } \Phi(0) = 0$$

This function has been studied in detail. Tables of its values for various values of x have been compiled. We shall see how this is done in Sec. 16.21 (Vol. II). Figs. 208 and 209 show the graph of the integrand $y = e^{-x^2}$ and the graph of the Laplace function $y = \Phi(x)$. That one of the antiderivatives

$$\int \sqrt{1-k^2\sin^2 x}\,dx + C \quad (k < 1)$$

---

* $\sqrt{1-\sin^2 z} = |\cos z|$. For the sake of definiteness, we only examine the case $|\cos z| = \cos z$.

which vanishes for $x=0$ is called an *elliptic integral* and is denoted by $E(x)$,

$$E(x)=\int \sqrt{1-k^2\sin^2 x}\,dx+C_2 \text{ if } E(0)=0$$

Tables of the values of this function have also been compiled for various values of x.

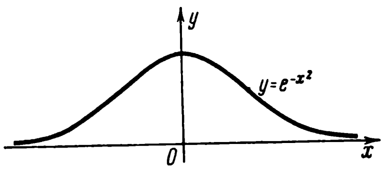

Fig. 208

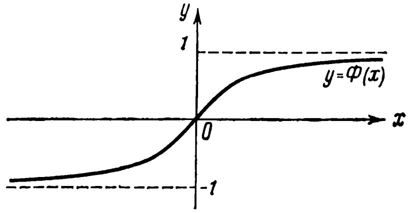

Fig. 209

Exercises on Chapter 10

I. Compute the integrals: **1.** $\int x^5\,dx$. *Ans.* $\frac{x^6}{6}+C$. **2.** $\int (x+\sqrt{x})\,dx$. *Ans.* $\frac{x^2}{2}+\frac{2x\sqrt{x}}{3}+C$. **3.** $\int\left(\frac{3}{\sqrt{x}}-\frac{x\sqrt{x}}{4}\right)dx$. *Ans.* $6\sqrt{x}-\frac{1}{10}x^2\sqrt{x}+C$. **4.** $\int\frac{x^2\,dx}{\sqrt{x}}$. *Ans.* $\frac{2}{5}x^2\sqrt{x}+C$. **5.** $\int\left(\frac{1}{x^2}+\frac{4}{x\sqrt{x}}+2\right)dx$. *Ans.* $-\frac{1}{x}-\frac{8}{\sqrt{x}}+2x+C$. **6.** $\int\frac{dx}{\sqrt[4]{x}}$. *Ans.* $\frac{4}{3}\sqrt[4]{x^3}+C$. **7.** $\int\left(x^2+\frac{1}{\sqrt[3]{x}}\right)^2 dx$. *Ans.* $\frac{x^5}{5}+\frac{3}{4}x^2\sqrt[3]{x^2}+3\sqrt[3]{x}+C$.

Integration by substitution: **8.** $\int e^{5x}\,dx$. *Ans.* $\frac{1}{5}e^{5x}+C$. **9.** $\int\cos 5x\,dx$. *Ans.* $\frac{\sin 5x}{5}+C$. **10.** $\int\sin ax\,dx$. *Ans.* $-\frac{\cos ax}{a}+C$. **11.** $\int\frac{\ln x}{x}\,dx$. *Ans.* $\frac{1}{2}\ln^2 x+C$. **12.** $\int\frac{dx}{\sin^2 3x}$. *Ans.* $-\frac{\cot 3x}{3}+C$. **13.** $\int\frac{dx}{\cos^2 7x}$. *Ans.* $\frac{\tan 7x}{7}+C$. **14.** $\int\frac{dx}{3x-7}$. *Ans.* $\frac{1}{3}\ln|3x-7|+C$. **15.** $\int\frac{dx}{1-x}$. *Ans.* $-\ln|1-x|+C$. **16.** $\int\frac{dx}{5-2x}$. *Ans.* $-\frac{1}{2}\ln|5-2x|+C$. **17.** $\int\tan 2x\,dx$. *Ans.* $-\frac{1}{2}\ln|\cos 2x|+C$. **18.** $\int\cot(5x-7)\,dx$. *Ans.* $\frac{1}{5}\ln|\sin(5x-7)|+C$. **19.** $\int\frac{dy}{\cot 3y}$. *Ans.* $-\frac{1}{3}\ln|\cos 3y|+C$. **20.** $\int\cot\frac{x}{3}\,dx$. *Ans.* $3\ln\left|\sin\frac{x}{3}\right|+C$. **21.** $\int\tan\varphi\cdot\sec^2\varphi\,d\varphi$. *Ans.* $\frac{1}{2}\tan^2\varphi+C$. **22.** $\int(\cot e^x)e^x\,dx$. *Ans.* $\ln|\sin e^x|+C$. **23.** $\int\left(\tan 4S-\cot\frac{S}{4}\right)dS$. *Ans.* $-\frac{1}{4}\ln|\cos 4S|-4\ln\left|\sin\frac{S}{4}\right|+C$.

24. $\int \sin^2 x \cos x\,dx$. *Ans.* $\frac{\sin^3 x}{3}+C$. 25. $\int \cos^3 x \sin x\,dx$. *Ans.* $-\frac{\cos^4 x}{4}+C$.

26. $\int \sqrt{x^2+1}\,x\,dx$. *Ans.* $\frac{1}{3}\sqrt{(x^2+1)^3}+C$. 27. $\int \frac{x\,dx}{\sqrt{2x^2+3}}$. *Ans.* $\frac{1}{2}\sqrt{2x^2+3}+C$. 28. $\int \frac{x^2\,dx}{\sqrt{x^3+1}}$. *Ans.* $\frac{2}{3}\sqrt{x^3+1}+C$.

29. $\int \frac{\cos x\,dx}{\sin^2 x}$. *Ans.* $-\frac{1}{\sin x}+C$. 30. $\int \frac{\sin x\,dx}{\cos^3 x}$. *Ans.* $\frac{1}{2\cos^2 x}+C$.

31. $\int \frac{\tan x}{\cos^2 x}dx$. *Ans.* $\frac{\tan^2 x}{2}+C$. 32. $\int \frac{\cot x}{\sin^2 x}dx$. *Ans.* $-\frac{\cot^2 x}{2}+C$.

33. $\int \frac{dx}{\cos^2 x\sqrt{\tan x-1}}$. *Ans.* $2\sqrt{\tan x-1}+C$. 34. $\int \frac{\ln(x+1)}{x+1}dx$. *Ans.* $\frac{\ln^2(x+1)}{2}+C$. 35. $\int \frac{\cos x\,dx}{\sqrt{2\sin x+1}}$. *Ans.* $\sqrt{2\sin x+1}+C$.

36. $\int \frac{\sin 2x\,dx}{(1+\cos 2x)^2}$. *Ans.* $\frac{1}{2(1+\cos 2x)}+C$. 37. $\int \frac{\sin 2x\,dx}{\sqrt{1+\sin^2 x}}$. *Ans.* $2\sqrt{1+\sin^2 x}+C$. 38. $\int \frac{\sqrt{\tan x+1}}{\cos^2 x}dx$. *Ans.* $\frac{2}{3}\sqrt{(\tan x+1)^3}+C$.

39. $\int \frac{\cos 2x\,dx}{(2+3\sin 2x)^3}$. *Ans.* $-\frac{1}{12}\frac{1}{(2+3\sin 2x)^2}+C$. 40. $\int \frac{\sin 3x\,dx}{\sqrt[3]{\cos^4 3x}}$. *Ans.* $\frac{1}{\sqrt[3]{\cos 3x}}+C$. 41. $\int \frac{\ln^2 x\,dx}{x}$. *Ans.* $\frac{\ln^3 x}{3}+C$. 42. $\int \frac{\arcsin x\,dx}{\sqrt{1-x^2}}$. *Ans.* $\frac{\arcsin^2 x}{2}+C$. 43. $\int \frac{\arctan x\,dx}{1+x^2}$. *Ans.* $\frac{\arctan^2 x}{2}+C$.

44. $\int \frac{\arccos^2 x}{\sqrt{1-x^2}}dx$. *Ans.* $-\frac{\arccos^3 x}{3}+C$. 45. $\int \frac{\operatorname{arccot} x}{1+x^2}dx$. *Ans.* $-\frac{\operatorname{arccot}^2 x}{2}+C$.

46. $\int \frac{x\,dx}{x^2+1}$. *Ans.* $\frac{1}{2}\ln(x^2+1)+C$. 47. $\int \frac{x+1}{x^2+2x+3}dx$. *Ans.* $\frac{1}{2}\ln(x^2+2x+3)+C$.

48. $\int \frac{\cos x\,dx}{2\sin x+3}$. *Ans.* $\frac{1}{2}\ln(2\sin x+3)+C$. 49. $\int \frac{dx}{x\ln x}$. *Ans.* $\ln|\ln x|+C$.

50. $\int 2x(x^2+1)^4\,dx$. *Ans.* $\frac{(x^2+1)^5}{5}+C$. 51. $\int \tan^4 x\,dx$. *Ans.* $\frac{\tan^3 x}{3}-\tan x+x+C$.

52. $\int \frac{dx}{(1+x^2)\arctan x}$. *Ans.* $\ln|\arctan x|+C$. 53. $\int \frac{dx}{\cos^2 x(3\tan x+1)}$. *Ans.* $\frac{1}{3}\ln|3\tan x+1|+C$. 54. $\int \frac{\tan^3 x}{\cos^2 x}dx$. *Ans.* $\frac{\tan^4 x}{4}+C$. 55. $\int \frac{dx}{\sqrt{1-x^2}\arcsin x}$. *Ans.* $\ln|\arcsin x|+C$. 56. $\int \frac{\cos 2x}{2+3\sin 2x}dx$. *Ans.* $\frac{1}{6}\ln|2+3\sin 2x|+C$.

57. $\int \cos(\ln x)\frac{dx}{x}$. *Ans.* $\sin(\ln x)+C$. 58. $\int \cos(a+bx)\,dx$. *Ans.* $\frac{1}{b}\sin(a+bx)+C$.

59. $\int e^{2x}\,dx$. *Ans.* $\frac{1}{2}e^{2x}+C$. 60. $\int e^{\frac{x}{3}}dx$. *Ans.* $3e^{\frac{x}{3}}+C$. 61. $\int e^{\sin x}\cos x\,dx$. *Ans.* $e^{\sin x}+C$. 62. $\int a^{x^2}x\,dx$. *Ans.* $\frac{a^{x^2}}{2\ln a}+C$. 63 $\int e^{\frac{x}{a}}dx$. *Ans.* $ae^{\frac{x}{a}}+C$.

64. $\int (e^{2x})^2\,dx$. *Ans.* $\frac{1}{4}e^{4x}+C$. 65. $\int 3^x e^x\,dx$. *Ans.* $\frac{3^x e^x}{\ln 3+1}+C$. 66. $\int e^{-3x}\,dx$. *Ans.* $-\frac{1}{3}e^{-3x}+C$. 67. $\int (e^{5x}+a^{5x})\,dx$. *Ans.* $\frac{1}{5}\left(e^{5x}+\frac{a^{5x}}{\ln a}+C\right)$.

68. $\int e^{x^2+4x+3}(x+2)\,dx$. *Ans.* $\frac{1}{2}e^{x^2+4x+3}+C$. 69. $\int \frac{(a^x-b^x)^2}{a^x b^x}\,dx$. *Ans.* $\frac{\left(\frac{a}{b}\right)^x-\left(\frac{b}{a}\right)^x}{\ln a-\ln b}-2x+C$. 70. $\int \frac{e^x\,dx}{3+4e^x}$. *Ans.* $\frac{1}{4}\ln(3+4e^x)+C$.

71. $\int \frac{e^{2x}\,dx}{2+e^{2x}}$. *Ans.* $\frac{1}{2}\ln(2+e^{2x})+C$. 72. $\int \frac{dx}{1+2x^2}$. *Ans.* $\frac{1}{\sqrt{2}}\arctan(\sqrt{2}x)+C$.

73. $\int \frac{dx}{\sqrt{1-3x^2}}$. *Ans.* $\frac{1}{\sqrt{3}}\arcsin(\sqrt{3}x)+C$. 74. $\int \frac{dx}{\sqrt{16-9x^2}}$. *Ans.* $\frac{1}{3}\arcsin\frac{3x}{4}+C$. 75. $\int \frac{dx}{\sqrt{9-x^2}}$. *Ans.* $\arcsin\frac{x}{3}+C$. 76. $\int \frac{dx}{4+x^2}$. *Ans.* $\frac{1}{2}\arctan\frac{x}{2}+C$. 77. $\int \frac{dx}{9x^2+4}$. *Ans.* $\frac{1}{6}\arctan\frac{3x}{2}+C$. 78. $\int \frac{dx}{4-9x^2}$. *Ans.* $\frac{1}{12}\ln\left|\frac{2+3x}{2-3x}\right|+C$. 79. $\int \frac{dx}{\sqrt{x^2+9}}$. *Ans.* $\ln|x+\sqrt{x^2+9}|+C$.

80. $\int \frac{dx}{\sqrt{b^2x^2-a^2}}$. *Ans.* $\frac{1}{b}\ln|bx+\sqrt{b^2x^2-a^2}|+C$. 81. $\int \frac{dx}{\sqrt{b^2+a^2x^2}}$. *Ans.* $\frac{1}{a}\ln|ax+\sqrt{b^2+a^2x^2}|+C$. 82. $\int \frac{dx}{a^2x^2-c^2}$. *Ans.* $\frac{1}{2ac}\ln\left|\frac{ax-c}{ax+c}\right|+C$.

83. $\int \frac{x^2\,dx}{5-x^6}$. *Ans.* $\frac{1}{6\sqrt{5}}\ln\left|\frac{x^3+\sqrt{5}}{x^3-\sqrt{5}}\right|+C$. 84. $\int \frac{x\,dx}{\sqrt{1-x^4}}$. *Ans.* $\frac{1}{2}\arcsin x^2+C$.

85. $\int \frac{x\,dx}{x^4+a^4}$. *Ans.* $\frac{1}{2a^2}\arctan\frac{x^2}{a^2}+C$. 86. $\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$. *Ans.* $\arcsin e^x+C$.

87. $\int \frac{dx}{\sqrt{3-5x^2}}$. *Ans.* $\frac{1}{\sqrt{5}}\arcsin\sqrt{\frac{5}{3}}x+C$. 88. $\int \frac{\cos x\,dx}{a^2+\sin^2 x}$. *Ans.* $\frac{1}{a}\arctan\left(\frac{\sin x}{a}\right)+C$. 89. $\int \frac{dx}{x\sqrt{1-\ln^2 x}}$. *Ans.* $\arcsin(\ln x)+C$.

90. $\int \frac{\arccos x-x}{\sqrt{1-x^2}}\,dx$. *Ans.* $-\frac{1}{2}(\arccos x)^2+\sqrt{1-x^2}+C$.

91. $\int \frac{x-\arctan x}{1+x^2}\,dx$. *Ans.* $\frac{1}{2}\ln(1+x^2)-\frac{1}{2}(\arctan x)^2+C$. 92. $\int \frac{\sqrt{1+\ln x}}{x}\,dx$. *Ans.* $\frac{2}{3}\sqrt{(1+\ln x)^3}+C$. 93. $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$. *Ans.* $\frac{4}{3}\sqrt{(1+\sqrt{x})^3}+C$.

94. $\int \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{x}}}$. *Ans.* $4\sqrt{1+\sqrt{x}}+C$. 95. $\int \frac{e^x\,dx}{1+e^{2x}}$. *Ans.* $\arctan e^x+C$.

96. $\int \frac{\cos x\,dx}{\sqrt[3]{\sin^2 x}}$. *Ans.* $3\sqrt[3]{\sin x}+C$. 97. $\int \sqrt{1+3\cos^2 x}\,\sin 2x\,dx$. *Ans.* $-\frac{2}{9}\sqrt{(1+3\cos^2 x)^3}+C$. 98. $\int \frac{\sin 2x\,dx}{\sqrt{1+\cos^2 x}}$. *Ans.* $-2\sqrt{1+\cos^2 x}+C$.

99. $\int \frac{\cos^3 x}{\sin^4 x}dx$. *Ans.* $\frac{1}{\sin x}-\frac{1}{3\sin^3 x}+C$. 100. $\int \frac{\sqrt[3]{\tan^2 x}}{\cos^2 x}dx$. *Ans.* $\frac{3}{5}\sqrt[3]{\tan^5 x}+C$.

101. $\int \frac{dx}{2\sin^2 x+3\cos^2 x}$. *Ans.* $\frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{2}{3}}\tan x\right)+C$.

Integrals of the form $\int \frac{Ax+B}{ax^2+bx+c}dx$: 102. $\int \frac{dx}{x^2+2x+5}$. *Ans.* $\frac{1}{2}\arctan\frac{x+1}{2}+C$.

103. $\int \frac{dx}{3x^2-2x+4}$. *Ans.* $\frac{1}{\sqrt{11}}\arctan\frac{3x-1}{\sqrt{11}}+C$. 104. $\int \frac{dx}{x^2+3x+1}$.
Ans. $\frac{1}{\sqrt{5}}\ln\left|\frac{2x+3-\sqrt{5}}{2x+3+\sqrt{5}}\right|+C$. 105. $\int \frac{dx}{x^2-6x+5}$. *Ans.* $\frac{1}{4}\ln\left|\frac{x-5}{x-1}\right|+C$.

106. $\int \frac{dz}{2z^2-2z+1}$. *Ans.* $\arctan(2z-1)+C$. 107. $\int \frac{dx}{3x^2-2x+2}$.
Ans. $\frac{1}{\sqrt{5}}\arctan\frac{3x-1}{\sqrt{5}}+C$. 108. $\int \frac{(6x-7)\,dx}{3x^2-7x+11}$. *Ans.* $\ln|3x^2-7x+11|+C$.

109. $\int \frac{(3x-2)\,dx}{5x^2-3x+2}$. *Ans.* $\frac{3}{10}\ln(5x^2-3x+2)-\frac{11}{5\sqrt{31}}\arctan\frac{10x-3}{\sqrt{31}}+C$.

110. $\int \frac{3x-1}{x^2-x+1}dx$. *Ans.* $\frac{3}{2}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+C$.

111. $\int \frac{7x+1}{6x^2+x-1}dx$. *Ans.* $\frac{2}{3}\ln(3x-1)+\frac{1}{2}\ln(2x+1)+C$. 112. $\int \frac{2x-1}{5x^2-x+2}dx$.
Ans. $\frac{1}{5}\ln(5x^2-x+2)-\frac{8}{5\sqrt{39}}\arctan\frac{10x-1}{\sqrt{39}}+C$.

113. $\int \frac{6x^4-5x^3+4x^2}{2x^2-x+1}dx$. *Ans.* $x^3-\frac{x^2}{2}+\frac{1}{4}\ln|2x^2-x+1|+\frac{1}{2\sqrt{7}}\arctan\frac{4x-1}{\sqrt{7}}+C$.

114. $\int \frac{dx}{2\cos^2 x+\sin x\cos x+\sin^2 x}$. *Ans.* $\frac{2}{\sqrt{7}}\arctan\frac{2\tan x+1}{\sqrt{7}}+C$.

Integrals of the form $\int \frac{Ax+B}{\sqrt{ax^2+bx+C}}dx$: 115. $\int \frac{dx}{\sqrt{2-3x-4x^2}}$.
Ans. $\frac{1}{2}\arcsin\frac{8x+3}{\sqrt{41}}+C$. 116. $\int \frac{dx}{\sqrt{1+x+x^2}}$. *Ans.* $\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+C$.

117. $\int \frac{dS}{\sqrt{2aS+S^2}}$. *Ans.* $\ln|S+a+\sqrt{2aS+S^2}|+C$.

118. $\int \frac{dx}{\sqrt{5-7x-3x^2}}$. *Ans.* $\frac{1}{\sqrt{3}}\arcsin\frac{6x+7}{\sqrt{109}}+C$. 119. $\int \frac{dx}{\sqrt{x(3x+5)}}$.
Ans. $\frac{1}{\sqrt{3}}\ln|6x+5+\sqrt{12x(3x+5)}|+C$. 120. $\int \frac{dx}{\sqrt{2-3x-x^2}}$.
Ans. $\arcsin\frac{2x+3}{\sqrt{17}}+C$. 121. $\int \frac{dx}{\sqrt{5x^2-x-1}}$.
Ans. $\frac{1}{\sqrt{5}}\ln|10x-1+\sqrt{20(5x^2-x-1)}|+C$. 122. $\int \frac{2ax+b}{\sqrt{ax^2+bx+C}}dx$.

Ans. $2\sqrt{ax^2+bx+c}+C$. **123.** $\int \frac{(x+3)\,dx}{\sqrt{4x^2+4x+3}}$.

Ans. $\frac{1}{4}\sqrt{4x^2+4x+3}+\frac{5}{4}\ln|2x+1+\sqrt{4x^2+4x+3}|+C$.

124. $\int \frac{(x-3)\,dx}{\sqrt{3+66x-11x^2}}$. *Ans.* $-\frac{1}{11}\sqrt{3+66x-11x^2}+C$.

125. $\int \frac{(x+3)\,dx}{\sqrt{3+4x-4x^2}}$. *Ans.* $-\frac{1}{4}\sqrt{3+4x-4x^2}+\frac{7}{4}\arcsin\frac{2x-1}{2}+C$.

126. $\int \frac{3x+5}{\sqrt{x(2x-1)}}\,dx$. *Ans.* $\frac{3}{2}\sqrt{2x^2-x}+\frac{23}{4\sqrt{2}}\ln(4x-1+\sqrt{8(2x^2-x)})+C$.

II. Integration by parts:

127. $\int xe^x\,dx$. *Ans.* $e^x(x-1)+C$. **128.** $\int x\ln x\,dx$. *Ans.* $\frac{1}{2}x^2\left(\ln x-\frac{1}{2}\right)+C$.

129. $\int x\sin x\,dx$. *Ans.* $\sin x-x\cos x+C$. **130.** $\int \ln x\,dx$. *Ans.* $x(\ln x-1)+C$.

131. $\int \arcsin x\,dx$. *Ans.* $x\arcsin x+\sqrt{1-x^2}+C$. **132.** $\int \ln(1-x)\,dx$.

Ans. $-x-(1-x)\ln(1-x)+C$. **133.** $\int x^n\ln x\,dx$. *Ans.* $\frac{x^{n+1}}{n+1}\left(\ln x-\frac{1}{n+1}\right)+C$.

134. $\int x\arctan x\,dx$. *Ans.* $\frac{1}{2}[(x^2+1)\arctan x-x]+C$. **135.** $\int x\arcsin x\,dx$.

Ans. $\frac{1}{4}[(2x^2-1)\arcsin x+x\sqrt{1-x^2}]+C$. **136.** $\int \ln(x^2+1)\,dx$.

Ans. $x\ln(x^2+1)-2x+2\arctan x+C$. **137.** $\int \arctan\sqrt{x}\,dx$.

Ans. $(x+1)\arctan\sqrt{x}-\sqrt{x}+C$. **138.** $\int \frac{\arcsin\sqrt{x}}{\sqrt{x}}\,dx$.

Ans. $2\sqrt{x}\arcsin\sqrt{x}+2\sqrt{1-x}+C$. **139.** $\int \arcsin\sqrt{\frac{x}{x+1}}\,dx$.

Ans. $x\arcsin\sqrt{\frac{x}{x+1}}-\sqrt{x}+\arctan\sqrt{x}+C$. **140.** $\int x\cos^2 x\,dx$.

Ans. $\frac{x^2}{4}+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+C$. **141.** $\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx$.

Ans. $x-\sqrt{1-x^2}\arcsin x+C$. **142.** $\int \frac{x\arctan x}{(x^2+1)^2}\,dx$.

Ans. $\frac{x}{4(1+x^2)}+\frac{1}{4}\arctan x-\frac{1}{2}\frac{\arctan x}{1+x^2}+C$. **143.** $\int x\arctan\sqrt{x^2-1}\,dx$.

Ans. $\frac{1}{2}x^2\arctan\sqrt{x^2-1}-\frac{1}{2}\sqrt{x^2-1}+C$. **144.** $\int \frac{\arcsin x}{x^2}\,dx$.

Ans. $\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|-\frac{1}{x}\arcsin x+C$. **145.** $\int \ln(x+\sqrt{1+x^2})\,dx$.

Ans. $x\ln|x+\sqrt{1+x^2}|-\sqrt{1+x^2}+C$. **146.** $\int \arcsin x\,\frac{x\,dx}{\sqrt{(1-x^2)^3}}$.

Ans. $\frac{\arcsin x}{\sqrt{1-x^2}}+\frac{1}{2}\ln\left|\frac{1-x}{1+x}\right|+C$.

Use trigonometric substitutions in the following examples:

147. $\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx$. *Ans.* $-\frac{\sqrt{a^2-x^2}}{x}-\arcsin\frac{x}{a}+C$. **148.** $\int x^2\sqrt{4-x^2}\,dx$. *Ans.* $2\arcsin\frac{x}{2}-\frac{1}{2}x\sqrt{4-x^2}+\frac{1}{4}x^3\sqrt{4-x^2}+C$. **149.** $\int \frac{dx}{x^2\sqrt{1+x^2}}$. *Ans.* $-\frac{\sqrt{1+x^2}}{x}+C$. **150.** $\int \frac{\sqrt{x^2-a^2}}{x}\,dx$. *Ans.* $\sqrt{x^2-a^2}-a\arccos\frac{a}{x}+C$.

151. $\int \frac{dx}{\sqrt{(a^2+x^2)^3}}$. *Ans.* $\frac{x}{a^2}\frac{1}{\sqrt{a^2+x^2}}+C$.

Integration of rational fractions:

152. $\int \frac{2x-1}{(x-1)(x-2)}\,dx$. *Ans.* $\ln\left|\frac{(x-2)^3}{x-1}\right|+C$. **153.** $\int \frac{x\,dx}{(x+1)(x+3)(x+5)}$. *Ans.* $\frac{1}{8}\ln\frac{(x+3)^6}{|x+5|^5\,|x+1|}+C$. **154.** $\int \frac{x^5+x^4-8}{x^3-4x}\,dx$. *Ans.* $\frac{x^3}{3}+\frac{x^2}{2}+4x+\ln\left|\frac{x^2(x-2)^5}{(x+2)^3}\right|+C$. **155.** $\int \frac{x^4\,dx}{(x^2-1)(x+2)}$. *Ans.* $\frac{x^2}{2}-2x+\frac{1}{6}\ln\frac{|x-1|}{|x+1|^3}+\frac{16}{3}\ln|x+2|+C$. **156.** $\int \frac{dx}{(x-1)^2(x-2)}$. *Ans.* $\frac{1}{x-1}+\ln\left|\frac{x-2}{x-1}\right|+C$. **157.** $\int \frac{x-8}{x^3-4x^2+4x}\,dx$. *Ans.* $\frac{3}{x-2}+\ln\frac{(x-2)^2}{x^2}+C$.

158. $\int \frac{3x+2}{x(x+1)^3}\,dx$. *Ans.* $\frac{4x+3}{2(x+1)^2}+\ln\frac{x^2}{(x+1)^2}+C$. **159.** $\int \frac{x^2\,dx}{(x+2)^2(x+4)^2}$. *Ans.* $-\frac{5x+12}{x^2+6x+8}+\ln\left(\frac{x+4}{x+2}\right)^2+C$. **160.** $\int \frac{dx}{x(x^2+1)}$. *Ans.* $\ln\frac{|x|}{\sqrt{x^2+1}}+C$.

161. $\int \frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}\,dx$. *Ans.* $\ln\frac{(x^2-2x+5)^{\frac{3}{2}}}{|x-1|}+\frac{1}{2}\arctan\frac{x-1}{2}+C$.

162. $\int \frac{x^3-6}{x^4+6x^2+8}\,dx$. *Ans.* $\ln\frac{x^2+4}{\sqrt{x^2+2}}+\frac{3}{2}\arctan\frac{x}{2}-\frac{3}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+C$.

163. $\int \frac{dx}{x^3+1}$. *Ans.* $\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+C$.

164. $\int \frac{3x-7}{x^3+x^2+4x+4}$. *Ans.* $\ln\frac{x^2+4}{(x+1)^2}+\frac{1}{2}\arctan\frac{x}{2}+C$. **165.** $\int \frac{4dx}{x^4+1}$. *Ans.* $\frac{1}{\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\sqrt{2}\arctan\frac{x\sqrt{2}}{1-x^2}+C$. **166.** $\int \frac{x^5}{x^3-1}\,dx$. *Ans.* $\frac{1}{3}[x^3+\ln(x^3-1)]+C$. **167.** $\int \frac{x^3+x-1}{(x^2+2)^2}\,dx$. *Ans.* $\frac{2-x}{4(x^2+2)}+\ln(x^2+2)^{\frac{1}{2}}-\frac{1}{4\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+C$. **168.** $\int \frac{(4x^2-8x)\,dx}{(x-1)^2(x^2+1)^2}$. *Ans.* $\frac{3x^2-x}{(x-1)(x^2+1)}+\ln\frac{(x-1)^2}{x^2+1}+\arctan x+C$. **169.** $\int \frac{dx}{(x^2-x)(x^2-x+1)^2}$. *Ans.* $\ln\left|\frac{x-1}{x}\right|-\frac{10}{3\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}-\frac{2x-1}{3(x^2-x+1)}+C$.

Integration of irrational functions:

170. $\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}\,dx$. *Ans.* $\frac{4}{3}\left[\sqrt[4]{x^3}-\ln\left(\sqrt[4]{x^3}+1\right)\right]+C$.

171. $\int \frac{\sqrt{x^3}-\sqrt[3]{x}}{6\sqrt[4]{x}}\,dx$. *Ans.* $\frac{2}{27}\sqrt[4]{x^9}-\frac{2}{13}\sqrt[12]{x^{13}}+C$. 172. $\int \frac{\sqrt[6]{x}+1}{\sqrt[6]{x^7}+\sqrt[4]{x^5}}\,dx$.

Ans. $-\frac{6}{\sqrt[6]{x}}+\frac{12}{\sqrt[12]{x}}+2\ln x-24\ln\left(\sqrt[12]{x}+1\right)+C$.

173. $\int \frac{2+\sqrt[3]{x}}{\sqrt[6]{x}+\sqrt[3]{x}+\sqrt{x}+1}\,dx$.

Ans. $\frac{6}{5}\sqrt[6]{x^5}-\frac{3}{2}\sqrt[3]{x^2}+4\sqrt{x}-6\sqrt[3]{x}+6\sqrt[6]{x}-9\ln\left(\sqrt[6]{x}+1\right)+$
$+\frac{3}{2}\ln\left(\sqrt[3]{x}+1\right)+3\arctan\sqrt[6]{x}+C$. 174. $\int \sqrt{\frac{1-x}{1+x}}\,\frac{dx}{x^2}$.

Ans. $\ln\left|\frac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1-x}-\sqrt{1+x}}\right|-\frac{\sqrt{1-x^2}}{x}+C$. 175. $\int \sqrt{\frac{1-x}{1+x}}\,\frac{dx}{x}$.

Ans. $2\arctan\sqrt{\frac{1-x}{1+x}}+\ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right|+C$. 176. $\int \frac{\sqrt[7]{x}+\sqrt{x}}{\sqrt[7]{x^8}+\sqrt[14]{x^{15}}}\,dx$.

Ans. $14\left[\sqrt[14]{x}-\frac{1}{2}\sqrt[7]{x}+\frac{1}{3}\sqrt[14]{x^3}-\frac{1}{4}\sqrt[7]{x^2}+\frac{1}{5}\sqrt[14]{x^5}\right]+C$.

177. $\int \sqrt{\frac{2+3x}{x-3}}\,dx$. *Ans.* $\sqrt{3x^2-7x-6}+\frac{11}{2\sqrt{3}}\times$
$\times\ln\left(x-\frac{7}{6}+\sqrt{x^2-\frac{7}{3}x-2}\right)+C$.

Integrals of the form $\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$:

178. $\int \frac{dx}{x\sqrt{x^2-x+3}}$. *Ans.* $\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2-x+3}-\sqrt{3}}{3}+\frac{1}{2\sqrt{3}}\right|+C$.

179. $\int \frac{dx}{x\sqrt{2+x-x^2}}$. *Ans.* $-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2+x-x^2}+\sqrt{2}}{x}+\frac{1}{2\sqrt{2}}\right|+C$.

180. $\int \frac{dx}{x\sqrt{x^2+4x-4}}$. *Ans.* $\frac{1}{2}\arcsin\frac{x-2}{x\sqrt{2}}+C$. 181. $\int \frac{\sqrt{x^2+2x}}{x}\,dx$.

Ans. $\sqrt{x^2+2x}+\ln\left|x+1+\sqrt{x^2+2x}\right|+C$. 182. $\int \frac{dx}{\sqrt{(2x-x^2)^3}}$.

Ans. $\frac{x-1}{\sqrt{2x-x^2}}+C$. 183. $\int \sqrt{2x-x^2}\,dx$.

Ans. $\frac{1}{2}\left[(x-1)\sqrt{2x-x^2}+\arcsin(x-1)\right]+C$. 184. $\int \frac{dx}{x-\sqrt{x^2-1}}$.

Ans $\frac{x^2}{2}+\frac{x}{2}\sqrt{x^2-1}-\frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+C$. 185. $\int \frac{dx}{(1+x)\sqrt{1+x+x^2}}$.

Ans. $\ln\left|\frac{x+\sqrt{1+x+x^2}}{2+x+\sqrt{1+x+x^2}}\right|+C$. **186.** $\int\frac{(x+1)}{(2x+x^2)\sqrt{2x+x^2}}\,dx$.

Ans. $-\frac{1}{\sqrt{2x+x^2}}+C$. **187.** $\int\frac{1-\sqrt{1+x+x^2}}{x\sqrt{1+x+x^2}}\,dx$.

Ans. $\ln\left|\frac{2+x-2\sqrt{1+x+x^2}}{x^2}\right|+C$. **188.** $\int\frac{\sqrt{x^2+4x}}{x^2}\,dx$.

Ans. $-\frac{8}{x+\sqrt{x^2+4x}}+\ln\left|x+2+\sqrt{x^2+4x}\right|+C$.

Integration of trigonometric functions:

189. $\int\sin^3 x\,dx$. *Ans.* $\frac{1}{3}\cos^3 x-\cos x+C$. **190.** $\int\sin^5 x\,dx$.

Ans. $-\cos x+\frac{2}{3}\cos^3 x-\frac{\cos^5 x}{5}+C$. **191.** $\int\cos^4 x\sin^3 x\,dx$.

Ans. $-\frac{1}{5}\cos^5 x+\frac{1}{7}\cos^7 x+C$. **192.** $\int\frac{\cos^3 x}{\sin^4 x}\,dx$. *Ans.* $\csc x-\frac{1}{3}\csc^3 x+C$.

193. $\int\cos^2 x\,dx$. *Ans.* $\frac{x}{2}+\frac{1}{4}\sin 2x+C$. **194.** $\int\sin^4 x\,dx$.

Ans. $\frac{3}{8}x-\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+C$. **195.** $\int\cos^6 x\,dx$.

Ans. $\frac{1}{16}\left(5x+4\sin 2x-\frac{\sin^3 2x}{3}+\frac{3}{4}\sin 4x\right)+C$. **196.** $\int\sin^4 x\cos^4 x\,dx$.

Ans. $\frac{1}{128}\left(3x-\sin 4x+\frac{\sin 8x}{8}\right)+C$. **197.** $\int\tan^3 x\,dx$. *Ans.* $\frac{\tan^2 x}{2}+\ln|\cos x|+C$.

198. $\int\cot^5 x\,dx$. *Ans.* $-\frac{1}{4}\cot^4 x+\frac{1}{2}\cot^2 x+\ln|\sin x|+C$. **199.** $\int\cot^3 x\,dx$.

Ans. $-\frac{\cot^2 x}{2}-\ln|\sin x|+C$. **200.** $\int\sec^8 x\,dx$.

Ans. $\frac{\tan^7 x}{7}+\frac{3\tan^5 x}{5}+\tan^3 x+\tan x+C$. **201.** $\int\tan^4 x\sec^4 x\,dx$.

Ans. $\frac{\tan^7 x}{7}+\frac{\tan^5 x}{5}+C$. **202.** $\int\frac{dx}{\cos^4 x}$. *Ans.* $\tan x+\frac{1}{3}\tan^3 x+C$.

203. $\int\frac{\cos x}{\sin^2 x}\,dx$. *Ans.* $C-\csc x$. **204.** $\int\frac{\sin^3 x\,dx}{\sqrt[3]{\cos^4 x}}$.

Ans. $\frac{3}{5}\cos^{\frac{5}{3}} x+3\cos^{-\frac{1}{3}} x+C$. **205.** $\int\sin x\sin 3x\,dx$. *Ans.* $-\frac{\sin 4x}{8}+\frac{\sin 2x}{4}+C$.

206. $\int\cos 4x\cos 7x\,dx$. *Ans.* $\frac{\sin 11x}{22}+\frac{\sin 3x}{6}+C$. **207.** $\int\cos 2x\sin 4x\,dx$.

Ans. $-\frac{\cos 6x}{12}-\frac{\cos 2x}{4}+C$. **208.** $\int\sin\frac{1}{4}x\cos\frac{3}{4}x\,dx$. *Ans.* $-\frac{\cos x}{2}+\cos\frac{1}{2}x+C$.

209. $\int\frac{dx}{4-5\sin x}$. *Ans.* $\frac{1}{3}\ln\left|\frac{\tan\frac{x}{2}-2}{2\tan\frac{x}{2}-1}\right|+C$. **210.** $\int\frac{dx}{5-3\cos x}$.

Ans. $\frac{1}{2}\arctan\left|2\tan\frac{x}{2}\right|+C$. **211.** $\int\frac{\sin x\,dx}{1+\sin x}$. *Ans.* $\frac{2}{1+\tan\frac{x}{2}}+x+C$.

212. $\int\frac{\cos x\,dx}{1+\cos x}$. *Ans.* $x-\tan\frac{x}{2}+C$. **213.** $\int\frac{\sin 2x}{\cos^4 x+\sin^4 x}\,dx$.

Ans. $\arctan(2\sin^2 x-1)+C$. **214.** $\int\frac{dx}{(1+\cos x)^2}$ *Ans.* $\frac{1}{2}\tan\frac{x}{2}+\frac{1}{6}\tan^3\frac{x}{2}+C$.

215. $\int\frac{dx}{\sin^2 x+\tan^2 x}$. *Ans.* $-\frac{1}{2}\left[\cot x+\frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)\right]+C$.

216. $\int\frac{\sin^2 x}{1+\cos^2 x}\,dx$. *Ans.* $\sqrt{2}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)-x+C$.