TD 01

Exercise 1

1. Solve in $\mathbb{R}^2$ the system

$$\begin{cases} (x-1)(y-2)=0 \\ (x+1)(y+3)=0 \end{cases}$$

2. Prove that the statement

$$\left(\forall n \in \mathbb{N}^* \setminus \{1\} : 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} \in \mathbb{N}\right) \text{ is false}$$

Exercise 2

1. Give negation of the following propositions :
   - **a)** $\forall x \in \mathbb{R}, \quad 2x > x.$
   - **b)** $\forall x \in \mathbb{R}, \quad x > 0 \Rightarrow \; 2x > x.$
   - **c)** $\exists n \in \mathbb{N}, \quad 5n+11 = 3n+14.$
   - **d)** $\forall \varepsilon \in \mathbb{R}, \quad (\varepsilon > 0) \Rightarrow (\exists n \in \mathbb{N}^* : \frac{1}{n} < \varepsilon)$
2. In the first three cases, indicate which one is true.

Exercise 3

Write the following sentences and negate them using quantifiers :

a) A : "For every two natural numbers a and b, there is a natural number c that satisfies $ac \neq bc$ or $a = b$."

b) B : "The sum of a rational number and an irrational number is an irrational number."

c) C : " If the sum and product of two real numbers belong to the set Q, then these two numbers belong to Q ".

d) D : " For each x and y of $\mathbb{R}$ we have $xy = 0$ is equivalent to $x = 0$ or $y = 0$."

Exercise 4

Are the following sentences true or false ? Justify your answer.

a) P_1 : "$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : x - y = 1$"

b) P_2 : "$\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : x - y = 1$"

c) P_3 : "$P_1 \Rightarrow P_2$ ".

d) P_4 : "$P_2 \Rightarrow P_1$ ".

Exercise 5

Let f function of $\mathbb{R}$ in $\mathbb{R}$. Translate the following expressions into quantifier terms :

1. f is bounded above.
2. f is bounded.
3. f is even.
4. f never equals zero.
5. f is periodic.
6. f is increasing.
7. f is not the zero function.
8. f attains all values in $\mathbb{N}$.

Exercise 6

Show by recurrence that :

1. $\forall n \in \mathbb{N}^\star : 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$
2. $\forall n \in \mathbb{N}$, $4^n + 6n - 1$ is a multiple of 9.

Exercise 7

By the absurd show that :

$\forall n \in \mathbb{N}$, n^2 even $\Rightarrow n$ is even.