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Physics 1

Mechanics of the material point

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Online courses

Mechanics of the material point

Chapter 1: Kinematics of a material point

- ✓ - **Movement Characteristics**
- ✓ - **Rectilinear Motion**
- ✓ - **Plane Motion**
- ✓ - **Movement in Space**
- ✓ - **Relative Motion**

1.Introduction

- ✓ **Kinematics analyzes the movement of "points" without considering the causes of motion.**

- ✓ **No discussion of forces or Newton's laws (purely mathematical).**

- ✓ **Material point: A body with negligible dimensions compared to distance traveled.**

Kinematic Magnitudes → Movement Characteristics

Reference System: Essential for analyzing movement two approaches:

1. Algebraic: Equation of motion along a trajectory

2. Vector: Vector analysis of motion

Ch# 1: kinematic

2. Position of the particle and reference frames

✓ To study motion we choose a reference frame (origin and axes).

✓ The position of a particle at time (t) is given by the position vector $\mathrm{r}(t)$.

✓ In Cartesian coordinates:

$$r(t) = x(t)i + y(t)j + z(t)k$$

The particle is in motion if at least one of $x(t), y(t), z(t)$ depends on t. If all are constant, the particle is at rest.

2. Mobile position

The position of a material point **M** at time **t** is represented in a orthonormal reference system **R** $(O,\vec{i},\vec{j},\vec{k})$ by a position vector $\overrightarrow{OM}$ (**See figure 1**).

$$\boxed{\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}}$$

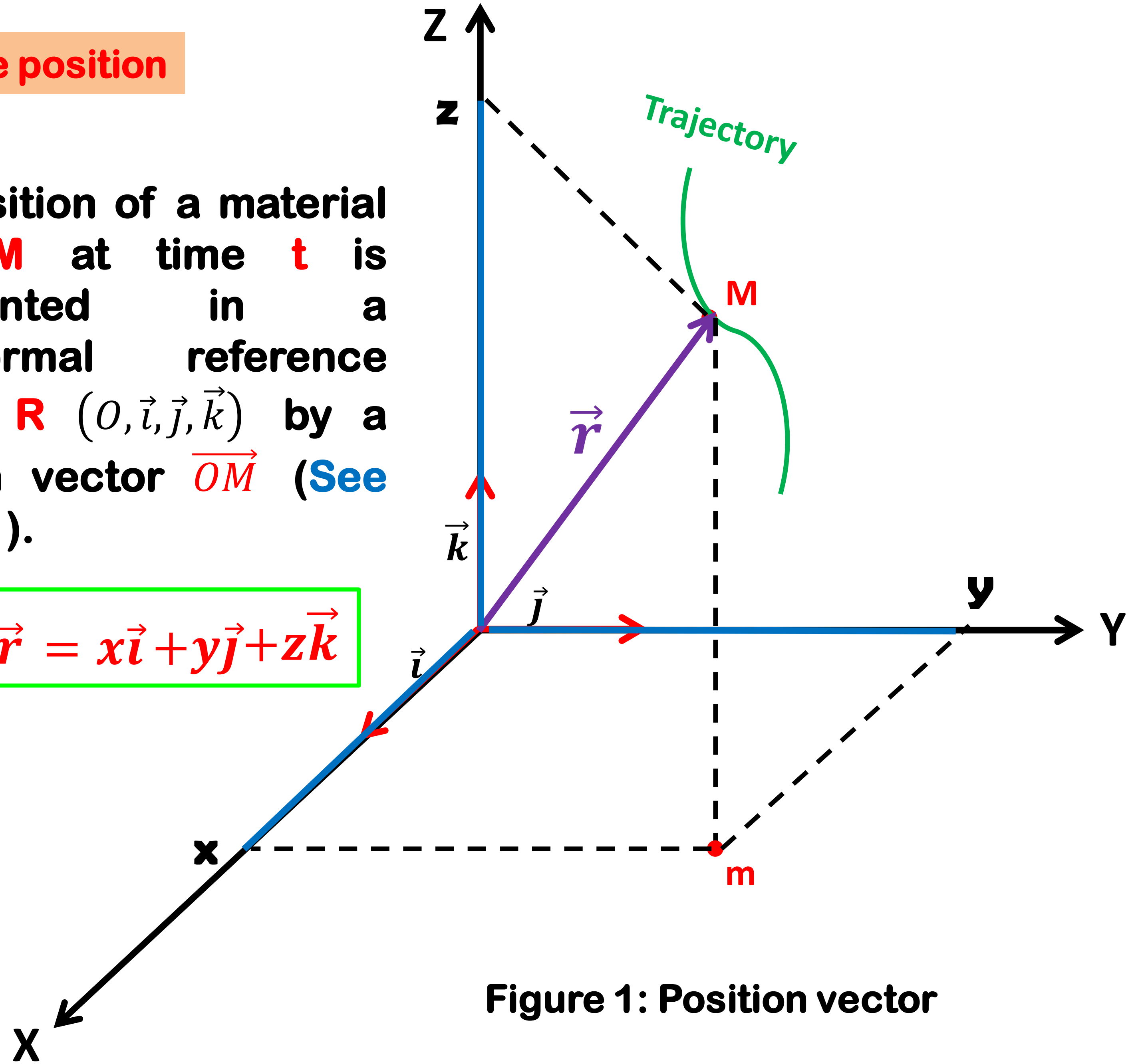


Figure 1: Position vector

The formula that expresses the position vector in Cartesian coordinates.

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \overrightarrow{OM}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Where (x, y, z) (Cartesian coordinates) are the components of the vector $\overrightarrow{OM}$ in the basis $(\vec{i}, \vec{j}, \vec{k})$

3. Time equations ⟷ Equations of Motion

Time (parametric) equations of motion

- **The time equations (parametric form) are:**

$$\boxed{x = x(t), y = y(t), z = z(t)}$$

- **These describe the coordinates of the particle as functions of time.**

- **The set** $(x, (t), y(t), z(t))$ **fully determines the motion.**

These functions are called the time equations of motion. They can be expressed in the form:

$$\boxed{x = f(t), y = g(t), z = h(t)}$$

4- Trajectory

•The trajectory is the geometric locus of points occupied by the particle over time.

•To find the Cartesian equation of the trajectory, eliminate the parameter "t" between $x(t), y(t), z(t)$.

Example (method): If $x = f(t)$and $y = g(t)$,solve $t = f^{-1}(x)$)or express t from one equation and substitute into the other to get $y = \Phi(x)$.

Example 1

We consider a material point M moving in space R $(O, \vec{i}, \vec{j}, \vec{k})$. The time equations of this movement describe the coordinates x(t), y(t) and z(t) of the point M as a function of time t . These equations are:

x= t+1 ; z=0 ; y= t^2+1

1/ Find the Cartesian equation of the trajectory, what is its form?

2/ Write the expression of the position vector at time t= 1s.

Solution:

x= t+1 .... (1) ; z=0 ....(2) ; y= t^2+1 ....(3)

1/ We take t from the equation x , which we replace by y:

$(1) \rightarrow$ x=t+1 ⇒ t=x-1 $(*)$

(*) → (3) ⇒ $y = (x-1)^2 + 1$ ⇒ $\boxed{y = x^2 - 2x + 2}$

so the trajectory described by point M is a parabolic trajectories.

Notes

The general equation of a parabola is:

$$y = ax^2 + bx + c$$

2. Expression of the position vector $\overrightarrow{OM}$ at time t= 1s

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

with $x = t+1$; $z=0$; $y = t^2+1$

$$\overrightarrow{OM} = \vec{r} = \underbrace{(t+1)}_{x}\vec{i} + \underbrace{(t^2+1)}_{y}\vec{j} + \underbrace{0}_{z}\vec{k}$$

With t=1s

$$\Rightarrow \quad \overrightarrow{OM} = \vec{r} = (1+1)\vec{i} + (1^2+1)\vec{j} + 0\vec{k}$$

$$\overrightarrow{OM} = \vec{r} = 2\vec{i} + 2\vec{j} + 0\vec{k} \quad \Rightarrow \quad \overrightarrow{OM} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

Example 1 :

- Find the Cartesian equation of the trajectory and $\mathbf{r}(1\ \mathbf{s})$.

Given :

$$x(t) = 2t, \qquad y(t) = t^2, \qquad z(t) = 0$$

Solution:

1. From $x(t) = 2t$**,** **we get** $t = \frac{x}{2}$

2. Substitute into $y(t)$**:**

$$y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$$

So the trajectory in the xy**- plane is the parabola** $y = \frac{x^2}{4}$

3 .Position vector at $t = 1$**s:**

$$\mathbf{r}(1) = x(1)\,\mathbf{i} + y(1)\,\mathbf{j} + z(1)\,\mathbf{k} = 2\mathbf{i} + 1\mathbf{j} + 0\mathbf{k}.$$

Example 2

The time equations of the material point M moving in space R $\left(O,\vec{i},\vec{j},\vec{k}\right)$ are:

$$x = t \;;\qquad y = 0 \;;\qquad z = -2t^2 + 2t$$

-what is the trajectory followed ?

Solution

$x = t$ …… (1) ; $y = 0$ …… (2); $z = -2t^2 + 2t$ …. (3)

- We take t from the equation x , which we replace by z:

$(1) \rightarrow t = x \Rightarrow (3) \Leftrightarrow \boxed{z = -2x^2 + 2x}$

so the trajectory described by point M is a parabolic trajectories.

Example 3

— Determine trajectory
Given:

$$x(t) = t, \qquad z(t) = -2t^2 + 2t$$

Solution:

Eliminate t: since $x = t$, substitute into z:

$$z = -2x^2 + 2x,$$

so the trajectory is the parabola $z = -2x^2 + 2x$ (in the xz-plane).

5-The velocity vector

Velocity is considered to be the distance traveled per unit of time.

5.1. Average velocity vector

The average velocity of a body that moves between two points M and M' is defined as the ratio between the displacement vector and the time interval in which the displacement takes place.

$$\vec{v}_{avg} = \frac{\overrightarrow{MM'}}{t_2 - t_1}$$

$$\overrightarrow{v_{avg}} // \overrightarrow{MM'}$$

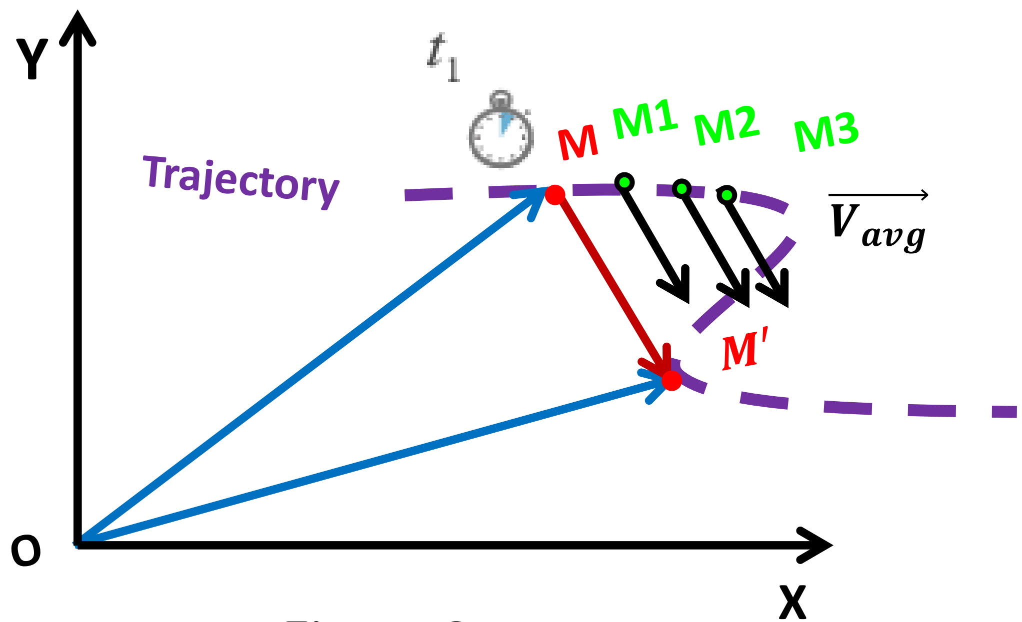


Figure 2

The average velocity vector is defined as follows

$$\vec{v}_{avg} = \frac{\overrightarrow{MM'}}{\Delta t} \quad \text{With } \Delta t = t_2 - t_1$$

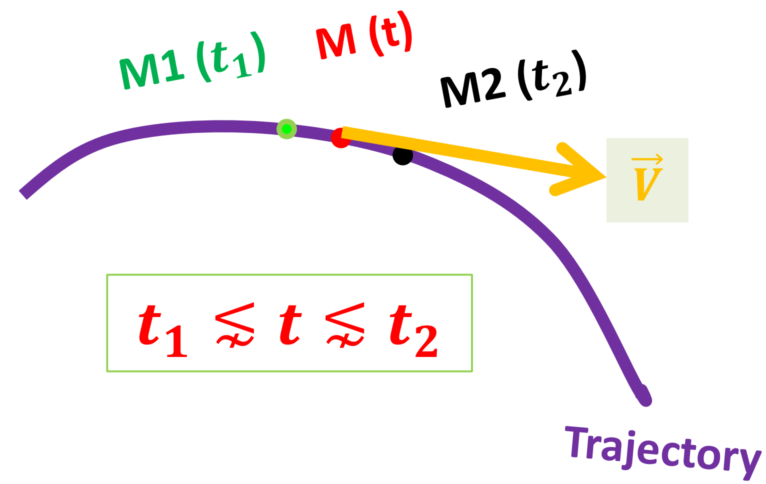
Where:

- $\overrightarrow{v_{avg}}$: Average velocity vector in the time studied.
- $\overrightarrow{MM'}$: Displacement vector in the time studied.
- $t_1,\ t_2$: Time in which the body is in the initial M and final M' points respectively

$$\overrightarrow{MM'} = \overrightarrow{OM'}(t_2) - \overrightarrow{OM}(t_1) = \Delta\overrightarrow{OM} \quad \Rightarrow \quad \vec{v}_{avg} = \frac{\Delta\overrightarrow{OM}}{\Delta t}$$

5.2. Instantaneous velocity vector

$$\vec{v}(t) = \frac{\overrightarrow{M_1M_2}}{\Delta t}$$ **With** $\Delta t = t_2 - t_1 \lll$



$$t_1 \lnsim t \lnsim t_2$$

$$\vec{v}(t) = \frac{\Delta\overrightarrow{OM}}{\Delta t}$$ **With** $\Delta t = t_2 - t_1 \lll$

we replace Δt by dt

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt}$$

The instantaneous velocity vector at time t is the derivative of the position vector $\overrightarrow{OM}$ with respect to time.

In the Cartesian coordinates for example, we deduce the expression of the instantaneous velocity vector from the expression of the position vector by deriving:

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$\vec{v} = \begin{pmatrix} \dot{x} = v_x \\ \dot{y} = v_y \\ \dot{z} = v_z \end{pmatrix}$$

Where

$$\dot{x} = \frac{dx}{dt};\ \dot{y} = \frac{dy}{dt};\ \dot{z} = \frac{dz}{dt}$$

$$\Rightarrow \vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

The instantaneous velocity vector $\vec{v}$ is carried by the tangent to the trajectory at point M; it is always oriented in the direction of movement.

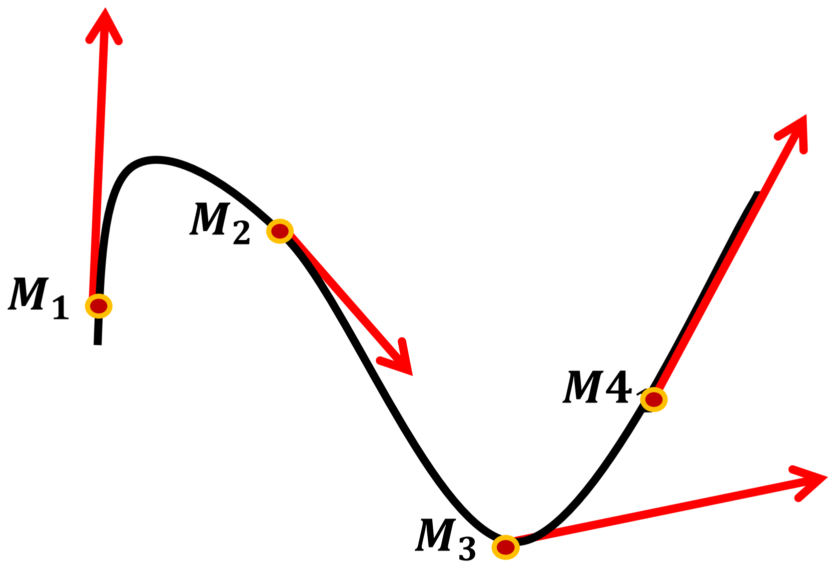


Figure 3: instantaneous velocity vector

Magnitude of the instantaneous velocity vector modulus

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

The SI unit of velocity is (m/s)

Application

We consider a mobile with position vector $\overrightarrow{OM} = 3t\vec{\imath} - 2t^2\vec{\jmath}$

1. **Calculate** $\vec{V}(t)$.
2. **Deduce its norm (magnitude) at date "t".**
3. **Calculate velocity at date t=2s.**

Solution

1. $$\overrightarrow{OM} = \overbrace{3t}^{x}\vec{\imath} - \overbrace{2t^2}^{y}\vec{\jmath} \Rightarrow \vec{V}(t) = \frac{d\overrightarrow{OM}}{dt} \Rightarrow \begin{cases} \checkmark\ \vec{v} = \dfrac{dx}{dt}\vec{\imath} + \dfrac{dy}{dt}\vec{\jmath} \\ \checkmark\ \vec{v} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} \end{cases}$$

$$\boxed{\vec{V}(t) = \underbrace{3}_{v_x = \dot{x}}\vec{\imath} - \underbrace{4t}_{v_y = \dot{y}}\vec{\jmath}}$$

NB:
$(f^m)' = mf^{m-1}$

2. $$V = \sqrt{\dot{x}^2 + \dot{y}^2} \Rightarrow V = \sqrt{3^2 + (-4t)^2} \Rightarrow \boxed{V = \sqrt{9 + 16t^2}}$$

3. $V = \sqrt{9 + 16t^2}$ **With t=2s.**

$$\Rightarrow V = \sqrt{9 + 16(2)^2} \Rightarrow \boxed{V = \sqrt{73} = 8.54\ (m/s)}$$

The Velocity Vector

•Average velocity between times t_1 and t_2:

$$\mathbf{v}_{\text{avg}} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}(t_2) - \mathbf{r}(t_1)}{t_2 - t_1}$$

•Instantaneous velocity:

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

•In Cartesian components:

$$\mathbf{v}(t) = \dot{x}(t)\,\mathbf{i} + \dot{y}(t)\,\mathbf{j} + \dot{z}(t)\,\mathbf{k}$$

where $\dot{x} = \frac{dx}{dt}$,etc.

•Speed (magnitude): $v(t) = \| \mathbf{v}(t) \| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ .Units: m/s.

6. The acceleration vector

We consider acceleration to be the change in velocity per unit time. The SI unit of acceleration is (m/s²)

6.1. Average acceleration vector

Considering two different times t_1 and t_1 corresponding to the position vectors $\overrightarrow{OM}$ and $\overrightarrow{OM'}$ and the instantaneous velocity vectors $\vec{v}$ and $\overrightarrow{v'}$ (see Figure 4)

$$\vec{a}_{avg} = \frac{\overrightarrow{v\prime}-\vec{v}}{t_2-t_1} = \frac{\Delta\vec{v}}{\Delta t}$$

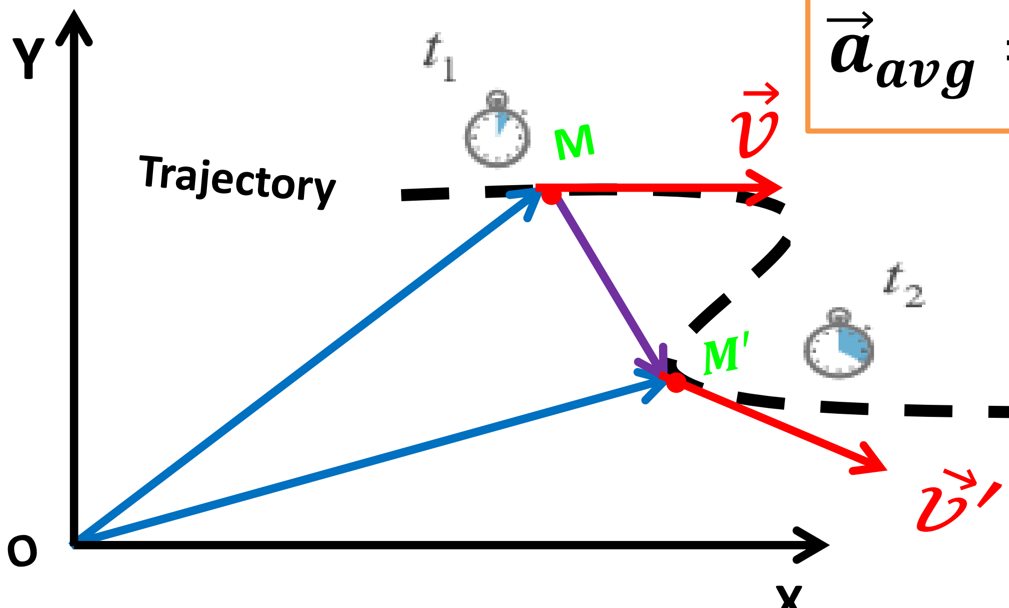


Figure 4

The average acceleration vector is defined by the expression:

$$\boxed{\vec{a}_{avg} = \frac{\vec{v\prime} - \vec{v}}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}}$$

6.2. Instantaneous acceleration vector

The instantaneous acceleration vector of a motion is defined as the derivative of the instantaneous velocity vector with respect to time.

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2}}$$

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} \Rightarrow \boxed{\vec{a} = \overbrace{\ddot{x}}^{a_x}\vec{i} + \overbrace{\ddot{y}}^{a_y}\vec{j} + \overbrace{\ddot{z}}^{a_z}\vec{k}}$$

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \Rightarrow \boxed{\vec{a} = \underbrace{\frac{d^2x}{dt^2}}_{a_x}\vec{i} + \underbrace{\frac{d^2y}{dt^2}}_{a_y}\vec{j} + \underbrace{\frac{d^2z}{dt^2}}_{a_z}\vec{k}}$$

Magnitude of the instantaneous acceleration vector

The magnitude of the acceleration is

$$a = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$$

The acceleration vector is always directed towards the inside of the curvature of the trajectory.

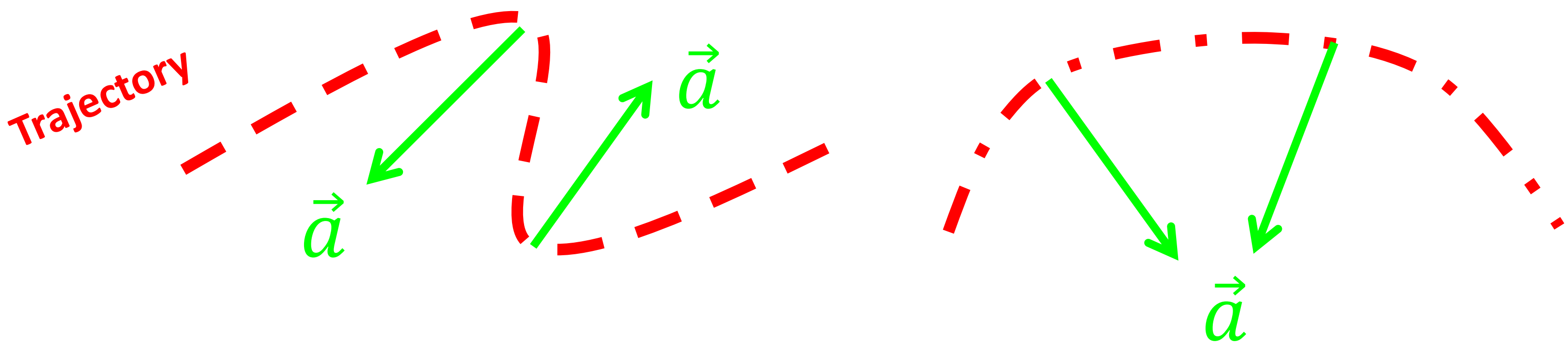


Figure 5: Acceleration vector

Notes

- The movement is accelerated (V↑) if ; $\vec{a} \cdot \vec{v} > 0$,
- The movement decelerated or retarded (V↓) if ; $\vec{a} \cdot \vec{v} < 0$.

Example 3

The position vector is $\overrightarrow{OM}\begin{pmatrix} 3t \\ 2t^3 + 1 \\ t^2 - 3 \end{pmatrix}$, deduce the instantaneous velocity vector and the acceleration vector, then calculate the magnitude of each of them.

Solution

$$\boxed{\overrightarrow{OM} = 3t\vec{\imath} + (2t^3 + 1)\vec{\jmath} + (t^2 - 3)\vec{k}}$$

$$\boxed{\vec{v} = 3\vec{\imath} + 6t^2\vec{\jmath} + 2t\vec{k}} \Rightarrow v = \sqrt{3^2 + (6t^2)^2 + (2t)^2} \Rightarrow \boxed{v = \sqrt{9 + 36t^4 + 4t^2}}$$

$$\boxed{\vec{a} = 0\vec{\imath} + 12t\vec{\jmath} + 2\vec{k}} \Rightarrow a = \sqrt{0^2 + (12t)^2 + 2^2} \Rightarrow \boxed{a = \sqrt{144t^2 + 4}}$$

Acceleration (average & instantaneous)

6. The Acceleration Vector

- **Average acceleration:**

$$\mathbf{a}_{\text{avg}} = \frac{\Delta \mathbf{v}}{\Delta t}$$

- **Instantaneous acceleration:**

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k}$$

- **Units: m/s^2.**
- **Note: acceleration points towards the center of curvature for curved motion components (centripetal component).**

Magnitude of acceleration: $\|\mathbf{a}(t)\| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$.

Remark: For planar motion, acceleration can be decomposed into tangential ($a_t = \dot{v}$)and normal ($a_n = \frac{v^2}{\rho}$) components, where ρ is radius of curvature.

Example 4

Original slide had "Example 3" but solution incomplete.

Example 3

— **Given:**

$$\mathbf{r}(t) = (3t^2)\,\mathbf{i} + (2t)\,\mathbf{j} + (t^3)\,\mathbf{k}$$

— **Find** $\mathbf{v}(t)$ **,** $\mathbf{a}(t)$ **,and** $v(t)$**at** $t = 1$**.**

Solution:

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = 6t\,\mathbf{i} + 2\,\mathbf{j} + 3t^2\,\mathbf{k}.$$
$$\mathbf{a}(t) = \ddot{\mathbf{r}}(t) = 6\,\mathbf{i} + 0\,\mathbf{j} + 6t\,\mathbf{k}.$$

At $t = 1$**:**

$$\mathbf{v}(1) = 6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}.$$

Speed at $t = 1$**:**

$$v(1) = \sqrt{6^2 + 2^2 + 3^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7 \textbf{ (units)}.$$