Chapter one: The set of real numbers

1 1.Algebraic structure of the set $\mathbb{R}$

The set of real numbers is a set that we denote by $\mathbb{R}$ equipped with the operation of addition and multiplication and an overall ordering relationship$\leq$checki the following Axiom.

A1) $\forall x, y, z \in \mathbb{R}: x + (y + z) = (x + y) + z.$

A2) $\forall x, y \in \mathbb{R}: x + y = y + x.$

A3) $\forall x \in \mathbb{R}: x + 0 = 0 + x = x.$

A4) $\forall x \in \mathbb{R}: x + (-x) = (-x) + x = 0.$

A5) $\forall x, y, z \in \mathbb{R}: x \cdot (y \cdot z) = (x \cdot y) \cdot z.$

A6) $\forall x, y \in \mathbb{R}: x \cdot y = y \cdot x.$

A7) $\forall x \in \mathbb{R}: x \cdot 1 = 1 \cdot x = x.$

A8) $\forall x \in \mathbb{R}^*: x \cdot x^{-1} = x^{-1} \cdot x = 1.$

A9) $\forall x, y, z \in \mathbb{R}: x \cdot (y + z) = x \cdot y + x \cdot z.$

A10) $\forall x \in \mathbb{R}: x \leq x.$

A11) $\forall x, y, z \in \mathbb{R}: \left(x \leq y \text{ و } y \leq z\right) \Rightarrow (x \leq z).$

A12) $\forall x, y \in \mathbb{R}: \left(x \leq y \text{ و } y \leq x\right) \Rightarrow (x = y).$

A13) $\forall x, y \in \mathbb{R}: x \leq y \text{ أو } y \leq x.$

A14) $\forall x, y, z \in \mathbb{R}: (x \leq y\ ) \Leftrightarrow (x + z \leq y + z).$

A15) $\begin{cases} \forall x, y \in \mathbb{R}; \forall z \in \mathbb{R}_+^*: (\ x \leq y\ ) \Leftrightarrow (\ x \cdot z \leq y \cdot z) \\ \forall x, y \in \mathbb{R}; \forall z \in \mathbb{R}_-^*: (\ x \leq y\ ) \Leftrightarrow (\ x \cdot z \geq y \cdot z) \end{cases}.$

Properties

1) $\forall x, y, x', y' \in \mathbb{R}: \left(x \leq y \text{و} x' \leq y'\right) \Rightarrow (x + x' \leq y + y').$

2) $\forall x, y, x', y' \in \mathbb{R}_+^*: \left(x \leq y \text{ و} x' \leq y'\right) \Rightarrow (x \cdot x' \leq y \cdot y').$

4) $\forall x, y, x', y' \in \mathbb{R}_+^*: (0 < x < y\ ) \Rightarrow \left(0 < \frac{1}{y} < \frac{1}{x}\right).$

1.2 Absolute value

Definition 1.1 let it be $x \in \mathbb{R}$

The absolute value of the real number x is the positive real number which we denote by $|x|$and defined as

$$|x| = \begin{cases} x, si\ x \geq 0 \\ -x, si\ x \leq 0 \end{cases}$$

Properties : $x. y\ r.$ is a real number where $r \geq 0$

1) $|x| \geq 0$; $|-x| = |x|$; $-|x| \leq x \leq |x|$

2) $|x| = 0 \iff x = 0$

3) $|x.y| = |x||y|$

4) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ $(y \neq 0)$

5) $|x + y| \leq |x| + |y|$

6) $|x| \leq r \iff -r \leq x \leq r$

7) .$|x| \geq r \iff x \leq -r$ or $x \geq r$

1.3.Limited parts from $\mathbb{R}$

Definition 1.2

Let A be a sub set of $\mathbb{R}$ and non-empty .

We say that A is bounded from above if and only if :

$$\exists b \in \mathbb{R} ;\ \forall x \in A : x \leq b$$

We say that A is bounded from below if and only if

$$\exists a \in \mathbb{R} ;\ \forall x \in A : x \geq a$$

A is bounded if and only if it is bounded from above and

Proposition 1.1 *The three following conditions are equivalent*

1*).A is bounded*

2) $\exists a \in \mathbb{R}$; $\exists b \in \mathbb{R} : \forall x \in A : a \leq x \leq b$.

3) $\exists M \in \mathbb{R}_+^*$; $\forall x \in A : |x| \leq M$

1.3.1 sup and inf.max and min

- *The smallest upper limit from A is called sup A*

The biggest lower limit from A is called inf A

If $supA \in A$ *it is called* $\max A$

If $infA \in A$ *it is called* $\min A$

Note

If A is infinite from above (from lowest, respectively) in $\mathbb{R}$ we write $supA = +\infty$

$(infA = -\infty$*, respectively).*

proposition 1.2

1)Let A be bounded from above, then

$$M = sup\ A \Leftrightarrow \begin{cases} \forall x \in A : x \leq M \\ \text{and} \\ \forall \varepsilon > 0\ ;\ \exists a \in A : M - \varepsilon < a \end{cases}$$

2) Let A be bounded from below, then

$$m = inf\ A \Leftrightarrow \begin{cases} \forall x \in A : x \geq m \\ \text{and} \\ \forall \varepsilon > 0\ ;\ \exists b \in A : m + \varepsilon > b \end{cases}$$

Proof

1) M is the smallest of the upper bounds if and only if the following proposition is false .

$$\exists M' < M; \forall x \in A : x \leq M'$$

Is true. So, if the proposition $\forall M' < M; \exists x \in A : x > M'$.

By putting $\varepsilon = M - M'$ *(*$\varepsilon > 0$*) so, the last proposition is written in the form:*

$$\forall \varepsilon > 0\ ;\ \exists x \in A : M - \varepsilon < x.$$

2) In the same way we prove the second case

Examples

1) $A = [1,2[$; $maxA =$ *unvailable ;supA =2 ; inf A=1* $minA = 1$

2) $\mathrm{A} = \{\frac{1}{n}\ ; n \in \mathbb{N}^*\}$

.$sup\ A = max\ A = 1$ فإن*then*$1 \in A \forall n \in \mathbb{N}^*: n \geq 1 \Rightarrow\ 0 < \frac{1}{n} \leq 1$

Let we proof that $infA = 0$

$$0 = infA \Leftrightarrow \begin{cases} \forall x \in A : x \geq 0 \\ \text{and} \\ \forall \varepsilon > 0\ ;\ \exists b \in A : 0 + \varepsilon > b \end{cases}$$

On the other side we have $\forall \varepsilon > 0\ ;\ \exists b \in A : 0 + \varepsilon > b \Leftrightarrow \forall \varepsilon > 0\ ;\ \exists n \in \mathbb{N}^* : \frac{1}{n} < \varepsilon.$

$and\ this\ last\ proposition\ is\ true\ and\ its\ according\ to\ archimed's\ axiom$

$$\forall \varepsilon > 0\ ;\ \exists n \in \mathbb{N}^* : \mathrm{n}\varepsilon > 1$$

$\min A = unvailable,$ because $0 \notin \mathrm{A}$.

1.3.2 Axiom of supermum and infimum:

*Any non-empty subset A of the real's $\mathbb{R}$ which is bounded above has a **supermum** in $\mathbb{R}$.*
*Any non-empty subset A of the real's $\mathbb{R}$ which is bounded below has a **infimum** in $\mathbb{R}$.*

1.4 Archimedean axiom

Theorem 1.1: $\forall x > 0;\ \forall y \in \mathbb{R}\ ; \exists n \in \mathbb{N}^*: y < nx.$

Proof:

We suppose that:

$\exists x > 0;\ \exists y \in \mathbb{R}\ ; \forall n \in \mathbb{N}^*: y \geq nx$ *or* $\exists x > 0;\ \exists y \in \mathbb{R}\ ; \forall n \in \mathbb{N}^*: n \leq \frac{y}{x}$,

that's mean the set $\mathbb{N}^*$*is limited from above it accepts an upper limit in* $\mathbb{R}$ *called M.*

So $\forall \varepsilon > 0\ ;\ \exists n_0 \in \mathbb{N}^*\ :\ M - \varepsilon < n_0$

by putting $\varepsilon = 1, we\ get\ the\ following\ : \exists n_0 \in \mathbb{N}^*\ :\ M - 1 < n_0$

or $\exists n_0 \in \mathbb{N}^*\ :\ M < n_0 + 1$

but $n_0 + 1 \in \mathbb{N}^*$

this is a contradiction because $supA = M$.

1.5 The integer part of a real number

For every real number x *there is only one integer which we denote as E(x)* or $[x]$ *it achives*

$E(x) \leq x < E(x) + 1$

$E(x)$ *is called the integer part of the real* x.

In other words $E(x)$ *is The largest integer less than or equal to* x.

<u>Examples</u>

1) $E(0\ ,1) = 0$ *since* $0 \leq 0{,}1 < 0 + 1$.
2) $E(-0\ ,1) = -1$ *since* $-1 \leq -0{,}1 < -1 + 1$.
3) $\forall n \in \mathbb{N}^*: E\left(\frac{1}{n+1}\right) = 0$ *since* $\forall n \in \mathbb{N}^*: 0 \leq \frac{1}{n+1} < 0 + 1$.

1.6 dense groups in $\mathbb{R}$

Theorem 1.2 *between every two different real numbers there is at least one rational number.*

Proof

Let y *and* x *be two real numbers where* $x < y$.

According to Archimedean axiom $\exists n \in \mathbb{N}^*: 1 < n(y - x)$ *or* $nx + 1 < ny$.

On the other hand we have $E(nx) \leq nx < E(nx) + 1$ *or*

$$nx < E(nx) + 1 \leq nx + 1 < ny.$$

So $nx < E(nx) + 1 < ny$ *then* $x < \frac{E(nx)+1}{n} < y$.

Well the rational number $\frac{(nx)+1}{n}$ *is bounded between the two real numbers* x, y.

definition 1.3

we denote the set of irrational numbers with I *or* Q^c

Theorem 1.3 *_between every two different real numbers there is at least one irrational number.*

To prove this theory we need the following two propositions

proposition *1.3* the number $\sqrt{2}$ *is an irrational number*

.**proposition** *1.4* if $x \in$ I and r ∈ Q ∗ then r$x \in$ I

Proof of the proposition 1.3

We suppose that $\sqrt{2} \in \mathbb{Q}$ *then there is only one duality of natural numbers (p,q) then* $\frac{p}{q} = \sqrt{2}$ *and* $gcd(p.q) = 1$*, then::*

we conclude that q^2 *divide* p^2 *since* q^2 *and* p^2 *prime* $\frac{p}{q} = \sqrt{2} \Leftrightarrow p = q\sqrt{2} \Leftrightarrow p^2 = 2q^2$ *among themselves then* q^2 *devide* 1 *thats mean* $q = 1$ *substitung in the previous equality we get* $p^2 = 2$ *and this is a contradiction because there is no natural number squared equal 2 .*

Proof of the proposition1.4 *We assume* $x \in I$ *and* $r \in \mathbb{Q}^*$ *and that* $rx \in \mathbb{Q}$ *and from him:*

$$\left(\frac{1}{r} \in \mathbb{Q}^* or\ rx \in \mathbb{Q}\right) \Rightarrow \frac{1}{r} rx \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$$

This is a contradiction because $x \in I$*.*

Proof of the theorem 1.3

Let y, x *be a real numbers, where* $x < y$*, according to the theorem 1.2, there sexist a rational number* r $(r \neq 0)$ such that: $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$ or $x < r\sqrt{2} < y$ and according to propositions 1.3 and 1.4 we conclude that $r\sqrt{2}$ is a irrational number.

Corollary 1.1 The two sets $\mathbb{Q}$ and I is dense in $\mathbb{R}$.

1.7 Intervals in $\mathbb{R}$

Let a, b *a real numbers, where* $a < b$*, we define*

$[a, b] = \{x \in \mathbb{R}: a \leq x \leq b\}$ *is called closed interval.*

$]a, b[= \{x \in \mathbb{R}: a < x < b\}$ *is called open interval.*

$[a, b[= \{x \in \mathbb{R}: a \leq x < b\}$ *is called half open interval.*

$]a, b] = \{x \in \mathbb{R}: a < x \leq b\}$ "

$[a, +\infty[= \{x \in \mathbb{R}: x \geq a\}$ *unbounded closed interval.*

$]-\infty, b] = \{x \in \mathbb{R}: x \leq b\}$ "

$]a, +\infty[= \{x \in \mathbb{R}: x > a\}$ *unbounded open interval.*

$]-\infty, b[= \{x \in \mathbb{R}: x < b\}$ "

$\mathbb{R} =]-\infty, +\infty[$ "

<u>Theorem 1.4</u>

The non-empty subset I of $\mathbb{R}$ *is an interval if and only if the following property is satisfied:*

$$\forall a, b \in I\ (a \leq b);\ \forall x \in \mathbb{R}: a \leq x \leq b \Rightarrow x \in I$$

Proof

(⇐)Necessary condition: It is a clear that: if the set I is a interval, then the property is true.

(⇒)Sufficient condition: If the property is true, then the set I is a interval.

We have four possible cases, case 1: I is bounded, case 2: I is bounded from above and not bounded from below, case 3: I is bounded from below and not bounded from above, case 4: I is neither bounded from above nor from below.

Let us prove that in the first case: either $I = [a,b]$ or $I = [a,b[$ or $I =]a,b]$ or $I =]a,b[$ where $a = inf\ I$ and $b = sup\ I$.

We have: $$b = sup\ I \Leftrightarrow \begin{cases} \forall x \in I : x \leq b \\ \text{و} \\ \forall \varepsilon > 0\ ;\ \exists b' \in I : b - \varepsilon < b' \dots\dots (1) \end{cases}.$$

and

$$a = inf\ I \Leftrightarrow \begin{cases} \forall x \in I : x \geq a \\ \text{و} \\ \forall \delta > 0\ ;\ \exists a' \in I : a + \delta > a' \dots\dots (2) \end{cases}.$$

case 1: *If* $a \in I$ *and* $b \in I$, *then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a \leq x \leq b \Rightarrow x \in [a,b] \Rightarrow I \subset [a,b]$$

$$\forall x \in \mathbb{R}: x \in [a,b] \Rightarrow a \leq x \leq b \Rightarrow x \in I \Rightarrow [a,b] \subset I$$

So

$$I = [a,b].$$

case 2: *If* $a \in I$ *and* $b \notin I$, *then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a \leq x < b \Rightarrow x \in [a,b[\Rightarrow I \subset [a,b[$$

$$\forall x \in \mathbb{R}: x \in [a,b[\Rightarrow a \leq x < b \Rightarrow b - x > 0$$

putting $\varepsilon = b - x$ *in* (1) *we get* $x < b'$ *and since* $a, b' \in I$, *then:*

$$a \leq x < b' \Rightarrow x \in I \Rightarrow [a,b[\subset I$$

so

$$I = [a,b[.$$

case 3: *If* $a \notin I$ *and* $b \in I$, *then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a < x \leq b \Rightarrow x \in]a,b] \Rightarrow I \subset]a,b]$$

$$\forall x \in \mathbb{R}: x \in]a,b] \Rightarrow a < x \leq b \Rightarrow x - a > 0$$

By putting $\delta = x - a$ *in (2) we get* $x > a'$ *and since* $a, a' \in I$, *then:*

$$a' < x \leq b \Rightarrow x \in I \Rightarrow]a,b] \subset I$$

So

$$I =]a, b]$$

case 4: *If* $a \notin I$ *and* $b \notin I$, *Then:*

$$\forall x \in \mathbb{R}: x \in I \Rightarrow a < x < b \Rightarrow x \in]a, b[\Rightarrow I \subset]a, b[$$

$$\forall x \in \mathbb{R}: x \in]a, b[\Rightarrow a < x < b \Rightarrow x - a > 0 \text{ and } b - x > 0.$$

By putting $\varepsilon = b - x$ *in (1) and* $\delta = x - a$ *in (2) we get* $x < b'$ *and* $a' < x$, *since* $a', b' \in I$, *then:*

$$a' < x \leq b' \Rightarrow x \in I \Rightarrow]a, b[\subset I.$$

So

$$I =]a, b[.$$

In the same way we prove that I is a interval in the other cases.