TP Session Report: Necessary and Sufficient Conditions for Optimality (Unconstrained Case)

Mathematics Department

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Objective of the Session

The goal of this session is to study the **first-order and second-order conditions** for identifying minima and maxima of differentiable functions without constraints. Students will use MATLAB to compute gradients, Hessians, and classify critical points.

Theoretical Background

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable.

- **First-order necessary condition:** If x^* is a local minimum, then

$$\nabla f(x^*) = 0.$$

- **Second-order necessary condition:** If x^* is a local minimum, then

$$H_f(x^*) \succeq 0.$$

- **Second-order sufficient condition:** If $\nabla f(x^*) = 0$ and $H_f(x^*) \succ 0$ (positive definite), then x^* is a **strict local minimum**.

MATLAB Commands

Gradient and Hessian

```
syms x y
f = x^2 + y^2 - 2*x*y + 4*x;
gradf = gradient(f, [x y])   % Gradient
Hf = hessian(f, [x y])       % Hessian
```

Checking Definiteness of the Hessian

```
eig(Hf)  % eigenvalues of Hessian
% if all positive -> positive definite
% if all nonnegative -> positive semidefinite
% if mixed signs -> saddle point
```

Numerical Minimization and Verification

```
f = @(x) x(1)^2 + x(2)^2 - 2*x(1)*x(2) + 4*x(1);
x0 = [1 1];
[xmin, fval] = fminunc(f, x0)
```

Exercises

1. Compute the gradient and Hessian of

$$f(x, y) = x^2 + y^2 - 2x + 3y.$$

   Find the critical point(s) and classify them.

2. For the function

$$f(x, y) = x^2 + y^2 - 2xy,$$

   compute the eigenvalues of the Hessian and interpret the result (minimum, maximum, or saddle point).

3. Use MATLAB `fminunc` to find the minimum of

$$f(x, y) = (x - 2)^2 + (y + 1)^2.$$

   Verify the sufficient conditions.

Conclusion

This TP session shows how to apply necessary and sufficient conditions for optimality using gradient and Hessian computations. MATLAB helps students classify critical points, confirm theoretical results, and visualize the nature of minima, maxima, or saddle points.

Keywords

First-order condition, Second-order condition, Gradient, Hessian, Critical point, Local minimum, Saddle point, MATLAB.