TP Session Report: Existence and Uniqueness of the Minimum of a Function (Without Constraints)

Mathematics Department

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Objective of the Session

The goal of this session is to study the existence and uniqueness of minima for real-valued functions without constraints. Students will use MATLAB to compute gradients, Hessians, and visualize functions in order to confirm theoretical results.

Theoretical Background

A function $f : \mathbb{R}^n \to \mathbb{R}$ has a local minimum at x^* if:

$$\nabla f(x^*) = 0, \quad \text{and} \quad H_f(x^*) \succeq 0,$$

where ∇f is the gradient and H_f is the Hessian.

- If the Hessian is **positive definite**, the critical point is a strict local minimum.
- If f is convex, the local minimum is also a global minimum.
- If f is strictly convex, the minimum is unique.

MATLAB Commands

Gradient and Hessian

```
syms x y
f = x^2 + y^2 + 2*x + 3*y;
gradf = gradient(f, [x y])   % Gradient
Hf = hessian(f, [x y])       % Hessian
```

Numerical Minimization

```
f = @(x) x(1)^2 + x(2)^2 + 2*x(1) + 3*x(2);
x0 = [0 0];                  % Initial guess
[xmin, fval] = fminunc(f, x0)
```

Visualization

```
f = @(x,y) x.^2 + y.^2 + 2*x + 3*y;
fsurf(f, [-5 5 -5 5])          % 3D plot
contourf(-5:0.1:5, -5:0.1:5, ...
   (meshgrid(-5:0.1:5)) )      % for level sets
```

Exercises

1. Compute the gradient and Hessian of $f(x, y) = x^2 + y^2 + 4x + 2y$. Verify that the function has a unique global minimum.
2. Use `fminunc` in MATLAB to find the minimum point of $f(x, y) = (x-1)^2 + (y+2)^2$. Compare with the exact analytical solution.
3. Plot the surface and contour of $f(x, y) = x^2 + y^2 + 2x + 3y$. Locate the minimum visually.

Conclusion

This TP session shows how theoretical conditions (gradient, Hessian, convexity) guarantee the existence and uniqueness of minima. MATLAB confirms these results numerically and graphically, giving students a complete picture of unconstrained optimization.

Keywords

Minimum, Gradient, Hessian, Convex function, Uniqueness, MATLAB, Unconstrained optimization.