

Chapter 3

Existence and Uniqueness Results

In this chapter, we present assumptions under which the problem

$$\inf_{x \in \mathbb{R}^n} f(x) \tag{3.1}$$

admits at least one solution, i.e.,

$$f(x^*) = \min_{x \in \mathbb{R}^n} f(x),$$

with $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$.

We define the domain of f as

$$\text{dom}(f) = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}.$$

We say that f is **proper** if it never takes the value $-\infty$ and $\text{dom}(f) \neq \emptyset$.

1. Extrema of a Function

A function f may have extremal values: minima (the smallest values) or maxima (the largest values), either on its entire domain of definition or on a certain subset.

Definition 3.0.1 (Definition 3.1). Let A be a subset of $\mathbb{R}^n$ and f a function from A to $\mathbb{R}$ (see Figure 3.1).

1. We say that f has a **global minimum** at the point $x^* \in A$ if

$$\forall x \in A, \quad f(x^*) \leq f(x). \tag{3.2}$$

2. We say that f has a **global maximum** at the point $x^* \in A$ if

$$\forall x \in A, \quad f(x^*) \geq f(x). \tag{3.3}$$

3. The minimum is called **strict** if

$$\forall x \in A, \quad x \neq x^*, \quad f(x^*) < f(x). \tag{3.4}$$

4. The maximum is called **strict** if

$$\forall x \in A, x \neq x^*, \quad f(x^*) > f(x). \quad (3.5)$$

5. We say that f has a **local minimum** at the point $x^* \in A$ if there exists $r > 0$ such that

$$\forall x \in B(r, x^*) \cap A, \quad f(x^*) \leq f(x). \quad (3.6)$$

6. We say that f has a **local maximum** at the point $x^* \in A$ if there exists $r > 0$ such that

$$\forall x \in B(r, x^*) \cap A, \quad f(x^*) \geq f(x). \quad (3.7)$$

7. We say that x^* is a **strict local minimum** of f on A if there exists $r > 0$ such that

$$\forall x \in B(r, x^*) \cap A, x \neq x^*, \quad f(x^*) < f(x). \quad (3.8)$$

8. We say that x^* is a **strict local maximum** of f on A if there exists $r > 0$ such that

$$\forall x \in B(r, x^*) \cap A, x \neq x^*, \quad f(x^*) > f(x). \quad (3.9)$$

Example 3.0.2 (3.1). The function $f(x) = x^2$ admits a strict minimum at the point $x^* = 0$ on $\mathbb{R}$.

Example 3.0.3 (3.2). The function $f(x) = \cos(x)$ admits global minima at the points $x^* = k\pi$ for $k \in \mathbb{Z}$ and k odd, and it admits global maxima at the points $x^* = k\pi$ for $k \in \mathbb{Z}$ and k even on $\mathbb{R}$.

Example 3.0.4 (3.3). The floor function $E(x)$ admits a non-strict local maximum at the point $x^* = 1$ on $\mathbb{R}$.

The following theorem shows that a local minimum of a convex function is also a global minimum:

Theorem 3.0.5 (3.1). *Let A be a convex subset of $\mathbb{R}^n$ and let f be a function defined on A . Then, if f admits a local minimum at some point $x^* \in A$, this point is also a global minimum of f on A .*

Proof

We proceed by contradiction. Suppose that the function f admits a local minimum at the point x^* which satisfies relation (3.6) such that x^* is not a global minimum, therefore

$$\exists x \in \mathbb{R}^n : f(x) \leq f(x^*). \quad (3.10)$$

It is clear that for any vector x of $B(r, x^*)$, and for a sufficiently small $\lambda \in]0, 1[$, we have

$$f(x) \leq f(x^* + \lambda(x - x^*)).$$

Let $y = x^* + \lambda(x - x^*)$, then $y \in B(r, x^*)$. Since f is convex, we have

$$f(y) = f(x^* + \lambda(x - x^*)) \leq (1 - \lambda)f(x^*) + \lambda f(x).$$

From inequality (3.10), we obtain

$$f(y) \leq (1 - \lambda)f(x^*) + \lambda f(x^*) = f(x^*).$$

This contradicts the assumption that x^* is a local minimum. □

Existence of an Extremum

In order to guarantee the existence and uniqueness of a global minimum, one must make assumptions about the objective function and the constraint set C . In fact, the existence of extrema is not guaranteed for every function, but for a continuous function on a compact set (closed and bounded), we have the following classical theorem of Weierstrass:

Theorem 3.0.6 (Weierstrass Theorem [?, ?, ?]). *Let C be a compact subset of $\mathbb{R}^n$ and $f : C \rightarrow \mathbb{R}$. If f is continuous on C , then f admits a global minimum and a global maximum on C , i.e.,*

$$\exists x^* \in C : \inf_{x \in C} f(x) = \min_{x \in C} f(x) = f(x^*),$$

and

$$\exists \hat{x} \in C : \sup_{x \in C} f(x) = \max_{x \in C} f(x) = f(\hat{x}).$$

Remark 3.0.7. On $\mathbb{R}$, the compact sets are the closed intervals $[a, b]$.

Example 3.0.8. Let f be a function defined on $C = \{x \in \mathbb{R}^2 : \|x\|^2 \leq 1\}$ by $f(x) = x_1 + x_2$. The set C is compact, therefore by the Weierstrass theorem, the function f is bounded (see Figure 3.2).

The search for an extremum is not limited to a closed and bounded set. In the case where $C = \mathbb{R}^n$, one cannot apply the Weierstrass theorem. Another useful concept for the existence of a global minimum is introduced in the following definition:

Definition 3.0.9. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$. The function f is said to be coercive if

$$\lim_{\|x\| \rightarrow \infty} f(x) = +\infty,$$

i.e., the function f grows large when $\|x\|$ is large.

Example 3.0.10. The function $f(x) = x^2$ is coercive.

Example 3.0.11. Another example of a coercive function is the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$f(x_1, x_2) = x_1^2 + x_2^2.$$

Example 3.0.12. Consider the function

$$f(x_1, x_2) = x_1^2 - x_2^2.$$

The function f is not coercive, because if we consider the sequence $x_n = (0, n)$ for $n \in \mathbb{N}$, we have

$$\lim_{n \rightarrow \infty} \|x_n\| = +\infty,$$

and

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} (-n^2) = -\infty.$$

Example 3.0.13. The function $f(x) = x^3$ is not coercive (since $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$).

Example 3.0.14. The function $f(x) = \exp(x)$ is not coercive (since $f(x) \rightarrow 0$ as $x \rightarrow -\infty$).

In the case where $C = \mathbb{R}^n$, we have the following result:

Theorem 3.0.15. *Let $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a proper, continuous, and coercive function. Then problem (3.1) admits at least one solution.*

Proof. Let $d = \inf_{x \in \mathbb{R}^n} f(x)$. Since f is proper, we have

$$d > -\infty. \tag{3.11}$$

From the definition of the infimum, we have

$$\forall \varepsilon > 0, \exists x(\varepsilon) : f(x(\varepsilon)) - \varepsilon \leq d.$$

For $\varepsilon_n = \frac{1}{n}$, we find

$$\forall n \geq 0, \exists x_n : f(x_n) - \frac{1}{n} \leq d,$$

which gives

$$d \leq f(x_n) \leq d + \frac{1}{n}.$$

The sequence $(x_n)_{n \in \mathbb{N}}$ is called a *minimizing sequence* and it satisfies

$$\lim_{n \rightarrow \infty} f(x_n) = d.$$

Now, we want to show that the sequence $(x_n)_{n \in \mathbb{N}}$ is bounded. Indeed, we argue by contradiction: suppose that $(x_n)_{n \in \mathbb{N}}$ is not bounded, then

$$\lim_{n \rightarrow \infty} \|x_n\| = +\infty.$$

The coercivity of f implies that

$$\lim_{n \rightarrow \infty} f(x_n) = +\infty,$$

which contradicts (3.11). Therefore, $(x_n)_{n \in \mathbb{N}}$ is bounded. Thus, we can extract from $(x_n)_{n \in \mathbb{N}}$ a convergent subsequence $(x_{n_k})_k$ such that

$$\lim_{k \rightarrow \infty} x_{n_k} = x^*.$$

Hence,

$$\lim_{k \rightarrow \infty} f(x_{n_k}) = f(x^*) = d.$$

Therefore, the problem admits a solution. □

Example 3.0.16. The function $f(x) = x^2$ is coercive and continuous on $\mathbb{R}$, therefore it admits a minimum at the point $x^* = 0$.

Example 3.0.17. The function $f(x) = x^3$ is not coercive and it does not admit a minimum on $\mathbb{R}$.

Example 3.0.18. The function

$$f(x) = -\exp\left(-\frac{x^2}{4}\right) \cos(x)$$

is not coercive but it admits a minimum at the point $x^* = 0$. Therefore, coercivity is a sufficient condition but not a necessary one in Theorem 3.3.

3.0.1 Uniqueness of an Extremum

An important concept for uniqueness is strict convexity.

Theorem 3.0.19. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$. If f is strictly convex, then problem (3.1) admits at most one solution.*

Proof. We prove by contradiction. Suppose there exist two global solutions x_1, x_2 with $x_1 \neq x_2$ (here $f(x_1) = f(x_2)$) for (3.1). Since f is strictly convex, for $\lambda = \frac{1}{2}$, we obtain

$$f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) < \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) = f(x_2),$$

which contradicts the fact that $f(x_2)$ is the smallest value. $\square$

Example 3.0.20. Let f be a function defined from $\mathbb{R}^2$ to $\mathbb{R}$, with

$$x = (x_1, x_2) \mapsto f(x) = x_1^2 + x_2^2.$$

By computing its Hessian matrix, we find

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \succeq 0,$$

which shows that f is strictly convex. Thus, f is strictly convex. From the results seen in the previous section, f is also proper and coercive, and therefore it admits a unique global minimum at the point $x^* = (0, 0)$.

Definition 3.0.21. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, with $f \in C^1$. We say that f is an **elliptic function** with constant $\alpha \geq 0$ if

$$\forall x, y \in \mathbb{R}^n : \quad \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \alpha \|x - y\|^2.$$

Theorem 3.0.22. *If the objective function f of problem (3.1) is elliptic and coercive, then f is strictly convex. In particular, problem (3.1) admits a unique optimal solution.*

3.1 Exercises

Exercise 3.1 Consider the following optimization problem:

$$\min_{x \in A} f(x). \quad (3.12)$$

Study existence and uniqueness of solutions of problem (3.12) under the following cases:

1. $f(x) = x^2$ and
 - a) $A = \{x \in \mathbb{R} : |x| \leq 1\}$.
 - b) $A = \{x \in \mathbb{R} : |x| \geq 1\}$.
 - c) $A = \{x \in \mathbb{R} : |x| = 1\}$.
 - d) $A = \{x \in \mathbb{R} : |x| > 1\}$.
2. $f(x) = x_1 - x_2$ and
 - a) $A = \{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}$.
 - b) $A = \{x \in \mathbb{R}^2 : \|x\|_2 \geq 1\}$.
 - c) $A = \{x \in \mathbb{R}^2 : x_1^2 \leq 1 - x_2\}$.
 - d) $A = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$.

Exercise 3.2 (Rayleigh inequality) Denote by $S_n(\mathbb{R})$ the set of real symmetric $n \times n$ matrices.

1. Show that if $H \in S_n(\mathbb{R})$ is positive definite then there exists a constant $\alpha > 0$ such that

$$\langle x, Hx \rangle \geq \alpha \|x\|^2, \quad \forall x \in \mathbb{R}^n.$$

2. Let $H \in S_n(\mathbb{R})$ be positive definite. Deduce that the quadratic function

$$f(x) = \frac{1}{2} \langle x, Hx \rangle - \langle b, x \rangle + c$$

is coercive.

3. Deduce that the optimization problem $\min_{x \in \mathbb{R}^n} f(x)$ admits a unique solution.

Hint 3.1. For $H \in S_n(\mathbb{R})$ recall that there exists an orthogonal matrix O whose columns are eigenvectors of H such that

$$H = O^\top D O,$$

where D is diagonal with the eigenvalues of H on the diagonal.

Exercise 3.3 Suppose x^* is a local minimum of f on A , and let $A_0 \subset A$ with $A_0 \neq \emptyset$.

1. Show that if x^* is an interior point of A (relative to $\mathbb{R}^n$), then x^* is also a local minimum of f on A_0 .
2. Show that the same conclusion need not hold if x^* is not an interior point of A .