4.4 Exercises

Exercise 4.4.1

Solve the following system :

$$S : \begin{cases} x + y + z = 3 \\ 2x + y + z = 2 \\ x + 2y + z = 1 \end{cases}$$

Exercise 4.4.2

Solve the following system :

$$S : \begin{cases} 3x + y - 2z + 3t = 0 \\ -x + 2y - 4z + 6t = 2 \\ 2x - y + 2z - 3t = 0 \end{cases}$$

Exercise 4.4.3

Depending on the values of a, solve the following systems :

$$S_1 : \begin{cases} ax + y = 2 \\ (a^2 + 1)x + 2ay = 1 \end{cases} \quad , \quad S_2 : \begin{cases} (a+1)x + (a-1)y = 1 \\ (a-1)x + (a+1)y = 1 \end{cases}$$

Exercise 4.4.4

Consider the system (S),

$$S : \begin{cases} x - my + m^2 z = 2m \\ mx - m^2 y + mz = 2m \\ mx + y - m^3 z = 1 - m \end{cases}$$

Solve (S), specifying the values of m for which it is a Cramer system.

4.5 Solution of exercises

Solution 4.4.1

$$S : \begin{cases} x + y + z = 3 \\ 2x + y + z = 2 \\ x + 2y + z = 1 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$det\ A = det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = 1 \neq 0, \ rg \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = n = p = 3.$$

Therefore, S is a Cramer system.

$$x = \frac{detA_1}{detA} = \frac{\begin{vmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}} = -1 \ , y = \frac{detA_2}{detA} = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}} = -2, \text{ and } \ z = \frac{detA_3}{detA} = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}} = 6$$

Solution 4.4.2

$$S:\begin{cases}3x+y-2z+3t=0\\-x+2y-4z+6t=2\\2x-y+2z-3t=0\end{cases}\Leftrightarrow\begin{pmatrix}3&1&-2&3\\-1&2&-4&6\\2&-1&2&3\end{pmatrix}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix}$$

The $rgA = rg\begin{pmatrix}3&1&-2&3\\-1&2&-4&6\\2&-1&2&3\end{pmatrix}=3$. Take $M=\begin{pmatrix}3&1\\-1&2\end{pmatrix}\Rightarrow \det M=7\neq 0$

Consider the following system :

$$\begin{cases}3x+y=2z-3t\\-x+2y=2+4z-6t\end{cases}\Rightarrow\begin{cases}x=\frac{-2}{7}\\y=2z\ -3t+\frac{6}{7}\end{cases}$$

Substitute into the third equation :

$2x-y+2z-3t=\frac{-4}{7}\ -2z+3t\ -\frac{6}{7}+2z\ -3t=\frac{-10}{7}\neq 0$

Therefore, the system has no solution.

Solution 4.4.3

For system (S_1)

$$S_1:\begin{cases}ax+y=2\\(a^2+1)x+2ay=1\end{cases}\Leftrightarrow\begin{pmatrix}a&1\\a^2+1&2a\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$$

$det\ A=det\begin{pmatrix}a&1\\a^2+1&2a\end{pmatrix}=a^2-1\neq 0\Rightarrow a\neq\pm 1.$

So, there is a unique solution if and only if $a\neq\pm1$.

Then :

$$x=\frac{detA_1}{detA}=\frac{\begin{vmatrix}2&1\\1&2a\end{vmatrix}}{\begin{vmatrix}a&1\\a^2+1&2a\end{vmatrix}}=\frac{4a-1}{a^2-1},\quad y=\frac{detA_2}{detA}=\frac{\begin{vmatrix}a&2\\a^2+1&1\end{vmatrix}}{\begin{vmatrix}a&1\\a^2+1&2a\end{vmatrix}}=\frac{-2a^2\ +a-2}{a^2-1}$$

• If $a=1$ then the system becomes : $\begin{cases}x+y=2\\2x+2y=1\end{cases}\Leftrightarrow\begin{cases}x+y=2\\x+y=\frac{1}{2}\end{cases}$

However, we cannot have both $x+y=2$ and $x+y=\frac{1}{2}$ at the same time. Therefore, there is no solution.

• If $a=-1$ then the system becomes : $\begin{cases}-x+y=2\\2x-2y=1\end{cases}\Leftrightarrow\begin{cases}-(x-y)=2\\x-y=\frac{1}{2}\end{cases}$ impossible, so there is no solution.

For system (S_2)

$$S_2:\begin{cases}(a+1)x+(a-1)y=1\\(a-1)x+(a+1)y=1\end{cases}\Leftrightarrow\begin{pmatrix}a+1&a-1\\a-1&a+1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

Here, the determinant is $\begin{vmatrix}a+1&a-1\\a-1&a+1\end{vmatrix}=(a+1)^2-(a-1)^2=4a$

If $a\neq 0$ then we find a unique solution (x,y) with Cramer's formula

$$x=\frac{detA_1}{detA}=\frac{\begin{vmatrix}1&a-1\\1&a+1\end{vmatrix}}{\begin{vmatrix}a+1&a-1\\a-1&a+1\end{vmatrix}}=\frac{1}{2a},\quad y=\frac{detA_2}{detA}=\frac{\begin{vmatrix}a+1&1\\a-1&1\end{vmatrix}}{\begin{vmatrix}a+1&a-1\\a-1&a+1\end{vmatrix}}=\frac{1}{2a}.$$

• If $a=0$ then there is no solution.

Solution 4.4.4

$$S:\begin{cases}x-my+m^2z=2m\\mx-m^2y+mz=2m\\mx+y-m^3z=1-m\end{cases}\Leftrightarrow\begin{pmatrix}1&-m&m^2\\m&-m^2&m\\m&1&-m^3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2m\\2m\\1-m\end{pmatrix}.$$

$det\ A=det\begin{pmatrix}1&-m&m^2\\m&-m^2&m\\m&1&-m^3\end{pmatrix}=m^5-m=m(m^4-1)\neq 0\Rightarrow m\neq 0$ and $m\neq\pm1$.

• If $m\neq 0,1,-1$, then (S) is a Cramer system and

$x=\frac{m(2m^2-3m+3)}{1+m^2}$, $y=\frac{(m-1)^2(2m+1)}{(1+m^2)(1+m)}$ and $z=\frac{2}{1+m}$.