Chapter 4

Solution of Systems of Equations

4.1 System of Equations

Definition 4.1.1

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
A system of n linear equations with p unknowns and coefficients in $\mathbb{K}$ is called :

$$S : \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1p}x_p = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2p}x_p = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3p}x_p = b_3 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{np}x_p = b_n \end{cases}$$

where $(x_j)_{j=1,..,p}$ are the unknowns, and (a_{ij}), $b_j \in IK$.

4.1.1 Matrix Representation

Definition 4.1.2

Let $A = (a_{ij})_{1\leq i\leq n,\ 1\leq j\leq p}$, $B = \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}$, and $X = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_p \end{pmatrix}$. The system (S) becomes

$$AX = B.$$

If f is a linear map from IK^p to IK^n with A being the matrix associated with f in the canonical bases, and if we denote $X = (x_1, ..., x_p)$ and $b = (b_1, ..., b_n)$, then the system (S) becomes $f(X) = B$.

4.1.2 Solution of the System $AX = B$

Definition 4.1.3

A solution of the system (S) is any element $X = (x_1, ..., x_p)$, satisfying the n equations of (S). This is equivalent to finding a vector X such that $AX = B$ or an element $X \in IK^p$ such that $f(X) = B$.

Example 4.1.4

$$\begin{cases} x+2y=1 \\ 3x-y=4 \\ x-y=-2 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (1,4,-2)$$

4.2 Rank of a System of Equations

The rank of a linear system is the rank of the matrix $(a_{ij})_{1\leq i\leq n, 1\leq j\leq p}$. If r is the rank of the linear system (S), then $r \leq n$ and $r \leq p$.

4.3 Cramer's Method

Definition 4.3.1

The system (S) is said to be of Cramer if $n = p = r$, meaning that (S) is a system of n equations with n unknowns and $detA \neq 0$.

Theorem 4.3.2

Every Cramer system has a solution given by :

$$X = A^{-1}B.$$

Theorem 4.3.3

In a Cramer system, the solution is given by the formulas :

$$x_i = \frac{detA_i}{detA}, \qquad i = 1, ..., n.$$

Where A_i is the reduced matrix of A, obtained by replacing the i-th column with the vector B.

Example 4.3.4

1. Cramer's formulas for a system of two equations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$ are as follows if the determinant satisfies $ad - bc \neq 0$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix} \Leftrightarrow \begin{cases} ax+by=e \\ cx+dy=f \end{cases} \Rightarrow x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad \text{and } y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

For this system, it gives

$$\begin{cases} 2x+y=1 \\ 3x+7y=-2 \end{cases}$$

$$x = \frac{\begin{vmatrix} 1 & 1 \\ -2 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix}} = \frac{9}{11}, \qquad y = \frac{\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix}} = \frac{-7}{11}$$

2. Cramer's formulas for a system of three equations $AX = B \Leftrightarrow \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} j \\ k \\ l \end{pmatrix}$ are as follows if the determinant satisfies $\det A \neq 0$:

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad \text{and } z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

For this system, it gives

$$\begin{cases} 2x - y + z = 1 \\ x + y - z = 2 \\ 3x + y + z = 3 \end{cases} \Leftrightarrow \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$x = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix}} = \frac{6}{6} = 1 \quad ,y = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix}} \frac{3}{6} = \frac{1}{2}, \quad \text{and} \quad z = \frac{\begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 3 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix}} \frac{-3}{6} = \frac{-1}{2}$$

4.3.1 Case when $\mathbf{n = p}$ and $\mathbf{r < n}$:

Consider a system of n equations with n unknowns, but $rgA < n$, i.e., $detA = 0$. In this case, extract a matrix M from A, knowing that it is the largest invertible square matrix, i.e., $detM \neq 0$, contained in A and of order r. This is called a submatrix, and the unknowns associated with M become principal unknowns, while the remaining $(n - r)$ unknowns become parameters or arbitrary values. Consider the following system :

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1r}x_r = b_1 - (a_{1(r+1)}x_{r+1} + \ldots + a_{1n}x_n) = b'_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2r}x_r = b_2 - (a_{1(r+1)}x_{r+1} + \ldots + a_{1n}x_n) = b'_2 \\ \ldots \\ a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rr}x_r = b_r - (a_{1(r+1)}x_{r+1} + \ldots + a_{1n}x_n) = b'_r \end{cases}$$

This last system is a Cramer system, so it has a unique solution $(x_1, ..., x_r)$ depending on $(x_{r+1}, ..., x_n)$. If this solution satisfies the remaining $(n - r)$ equations, then the overall system has infinitely many solutions. If, on the other hand, $(x_1, ..., x_r)$ does not satisfy any of the remaining $(n - r)$ equations, then the overall system has no solution.

Example 4.3.5

$$S_1 : \begin{cases} 3x - y + 2z = 3 \\ 2x + 2y + z = 2 \\ x - 3y + z = 1 \end{cases} \Leftrightarrow \begin{pmatrix} 3 & -1 & 2 \\ 2 & 2 & 1 \\ 1 & -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$detA = det \begin{pmatrix} 3 & -1 & 2 \\ 2 & 2 & 1 \\ 1 & -3 & 1 \end{pmatrix} = 0 \Rightarrow (S_1)$ is not a Cramer system since $det \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix} \neq 0 \Rightarrow rg(A) = 2$, and we consider x, y as unknowns and z as a parameter. Thus, we obtain the system :

$$\begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 - 2z \\ 2 - z \end{pmatrix} \Leftrightarrow \begin{cases} 3x - y = 3 - 2z \\ 2x + 2y = 2 - z \end{cases}$$

which is a Cramer system and has a unique solution (x, y) depending on z.

$$x = \frac{det \begin{pmatrix} 3 - 2z & -1 \\ 2 - z & 2 \end{pmatrix}}{det \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}} = \frac{8-5z}{8}, \; y = \frac{det \begin{pmatrix} 3 & 3 - 2z \\ 2 & 2 - z \end{pmatrix}}{det \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}} = \frac{z}{8}$$

Now, check if (x, y) satisfies $x - 3y + z = 1$ (remaining equation) : $\frac{8-5z}{8} - 3(\frac{z}{8}) + z = \frac{6}{8} \neq -5$. Therefore, the system has infinitely many solutions given by :
$\frac{(\frac{8-5z}{8}, \frac{z}{8}, z)}{z} \in \mathbb{R}$.

4.3.2 Case when $n \neq p$:

If the number of equations is not equal to the number of unknowns, determine the rank of A first and proceed as before. If M is a matrix contained in A and of order r with $detM \neq 0$, then consider the system of r equations with r unknowns corresponding to M, which is a Cramer system.

If the solution satisfies the remaining equations, then the overall system has infinitely many solutions. Otherwise, it has no solution.

Example 4.3.6

$$\begin{cases} 3x - y = 4 \\ 2x + 2y = 3 \\ x - 5y = -5 \end{cases} \Leftrightarrow \begin{pmatrix} 3 & -1 \\ 2 & 2 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -5 \end{pmatrix}$$

The rank of $\begin{pmatrix} 3 & -1 \\ 2 & 2 \\ 1 & -5 \end{pmatrix} \leq 2$, choose $M = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix} \Rightarrow detM = 8 \neq 0 \Rightarrow rg(M) = 2$. Take the system :

$$\begin{cases} 3x - y = 4 \\ 2x + 2y = 3 \end{cases} \Rightarrow \begin{cases} x = \frac{11}{8} \\ y = \frac{1}{8} \end{cases}$$

Now, check the remaining equation :
$x - 5y = -5 \Rightarrow \frac{11}{8} - 5(\frac{1}{8}) = \frac{6}{8} \neq -5$. Therefore, the system has no solution.