### Exercise 3.8.4

Consider the matrix $A$ defined as :

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

1. Find $a,\ b \in \mathbb{R}$ such that $A^2 = aI_3 + bA$.
2. Deduce that $A$ is invertible and determine its inverse.

### Exercise 3.8.5

Consider the matrix associated with the linear transformation $f$ defined on $\mathbb{R}^3$ with respect to the canonical basis :

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

1. Determine the linear application $f$.
2. Find $ker\ f$ and $Im\ f$, along with their dimensions. Is $f$ bijective ?
3. Let $S = \{v_1 = (1,1,1), v_2 = (1,0,1), v_3 = (2,-1,0)\}$ :
   (a) Show that $S$ is a basis for $\mathbb{R}^3$.
   (b) Find the matrix associated with $f$ with respect to the basis $S$.

### Exercise 3.8.6

Consider the matrix $A$ defined as :

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$

1. Determine the eigenvalues of $A$.
2. Show that $A$ is diagonalizable.
3. Find $P$ and calculate $A^k$.

## 3.9 Solution of exercises

### Solution 3.8.1

1. • $A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

   • $A^3 = A^2 \times A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$

   • $A^3 - A^2 + A - I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$

2. $A^3 - A^2 + A - I = O \Leftrightarrow A(A^2 - A + I) = I \quad$ donc $A^{-1} = A^2 - A + I$
3. $A^3 = A^2 - A + I$ donc $A^4 = A(A^2 - A + I) = A^3 - A^2 + A = (A^2 - A + I\ ) - A^2 + A = I$

## Solution 3.8.2

$$(A-2I)=\begin{pmatrix}3&0&1\\-1&3&-2\\-1&1&0\end{pmatrix}-2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&1\\-1&1&-2\\-1&1&-2\end{pmatrix}.$$

$$(A-2I)^2=\begin{pmatrix}1&0&1\\-1&1&-2\\-1&1&-2\end{pmatrix}\times\begin{pmatrix}1&0&1\\-1&1&-2\\-1&1&-2\end{pmatrix}=\begin{pmatrix}0&1&-1\\0&-1&1\\0&-1&1\end{pmatrix}.$$

$$(A-2I)^3=(A-2I)^2\times(A-2I)=\begin{pmatrix}0&1&-1\\0&-1&1\\0&-1&1\end{pmatrix}\times\begin{pmatrix}1&0&1\\-1&1&-2\\-1&1&-2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=O.$$

This implies that :
$A^3-3\times 2A^2+3\times 2^2A-2^3I=O$
Because $A$ and $I$ commute.
Which is equivalent to
$A^3-6A^2+12A-8I=O$
Or
$A^3-6A^2+12A=8I$
Then by dividing by 8 and factoring out $A$
$A(\frac{1}{8}A^2-\frac{3}{4}A+\frac{3}{2}I)=I$
This shows that $A$ is invertible and $A^{-1}=(\frac{1}{8}A^2-\frac{3}{4}A+\frac{3}{2}I)$

## Solution 3.8.3

1. $A$ is invertible if and only if $detA\neq 0$.

$$|A|=\left|\begin{pmatrix}5&6&-3\\-18&-19&9\\-30&-30&14\end{pmatrix}\right|\underset{=}{C_2:C_2-C_1}\left|\begin{pmatrix}5&1&-3\\-18&1&9\\-30&0&14\end{pmatrix}\right|\underset{=}{C_3:C_3+6C_2}\left|\begin{pmatrix}5&1&-3\\-18&1&9\\0&6&68\end{pmatrix}\right|$$

$$=\left|\begin{pmatrix}5&1&-3\\0&25&-45\\0&6&68\end{pmatrix}\right|=\left|\begin{pmatrix}5&1&-3\\0&1&-\frac{9}{5}\\0&6&68\end{pmatrix}\right|=\left|\begin{pmatrix}5&0&-\frac{6}{5}\\0&1&-\frac{9}{5}\\0&0&\frac{242}{5}\end{pmatrix}\right|=5\times 1\times\frac{242}{5}=242\neq 0.$$

Hence, $A$ is invertible.

2. Let's calculate $A^2-A-2I_3=0$.

$$A^2=A\cdot A=\begin{bmatrix}7&6&-3\\-18&-17&9\\-30&-30&16\end{bmatrix}$$

$$A^2-A=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}=2I$$

So, $A^2-A-2I_3=0$.
We observe that $A(A-I_3)=2I_3\Rightarrow A\left(\frac{1}{2}A-\frac{1}{2}I_3\right)=I_3\Rightarrow A^{-1}=\left(\frac{1}{2}A-\frac{1}{2}I_3\right)$.

## Solution 3.8.5

1. Find $a,\,b\in\mathbb{R}$ such that $A^2=aI_3+bA$.

$$A^2=A\cdot A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\cdot\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$$

$$=a\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+b\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}=\begin{bmatrix}a&b&b\\b&a&b\\b&b&a\end{bmatrix}$$

Therefore, $a=2$ and $b=1$.

2. $\det(A) = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 2 \neq 0$, hence $A$ is invertible.

$$A^2 = 2I_3 + A \Rightarrow A^2 - A = 2I_3 \Rightarrow A\left(\frac{1}{2}A - \frac{1}{2}I_3\right) = I_3 \Rightarrow A^{-1} = \frac{1}{2}A - \frac{1}{2}I_3.$$

**Solution 3.8.6**

1. Determine the function $f$.

$$f(x,y,z) = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x - y + 5z, 3x + 2z, x + y + 4z)$$

2. $\ker f = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = (0,0,0)\}$
So, $\dim \ker f = 0$ (since $f$ is injective).
$\operatorname{Im} f = \{f(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\}$

$$\operatorname{Im} f = \{x(1,3,1) + y(-1,0,1) + z(5,2,4) \mid x,y,z \in \mathbb{R}\}$$

The family $\{(1,3,1),(-1,0,1),(5,2,4)\}$ is linearly independent because $\det((1,3,1),(-1,0,1),(5,2,4)) = 23 \neq 0$. Therefore, $\dim \operatorname{Im} f = 3$, and $f$ is surjective, implying that $f$ is bijective.

3. Let $S = \{v_1 = (1,1,1), v_2 = (1,0,1), v_3 = (2,-1,0)\}$.
a) $S$ is a basis of $\mathbb{R}^3 \Leftrightarrow \det(v_1,v_2,v_3) \neq 0$

$$\det(v_1,v_2,v_3) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} \overset{C_3 = C_1 + C_3}{=} \begin{vmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 2 \neq 0$$

b) Let $A'$ be the matrix associated with $f$ using the basis $S$. $A' = P^{-1}AP$, with

$$P = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Let's perform a change of basis using $P^{-1} = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{-1}{2} & -1 & \frac{3}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$.

$$A' = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{-1}{2} & -1 & \frac{3}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 5 \\ 3 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 5 & -10 \\ \frac{7}{2} & 2 & -\frac{9}{2} \\ 8 & 5 & -8 \end{bmatrix}$$

**Solution**

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

1. Determine the eigenvalues of $A$, denoted as $\lambda \in \mathbb{R}$

$$P_A(\lambda) = |A - \lambda I_3| = \begin{vmatrix} 0-\lambda & 2 & -1 \\ 3 & -2-\lambda & 0 \\ -2 & 2 & 1-\lambda \end{vmatrix} = 0$$

The eigenvalues are 1, 2, and $-4$.

2. $A$ is diagonalizable because it has three distinct eigenvalues.
3. Find the eigenvectors.
   - For $\lambda = 1$

$$E_1 = \{v = (x, y, z) \in \mathbb{R}^3 \mid Av = v\} \Rightarrow \begin{cases} 2y - z = x \\ 3x - 2y = y \\ -2x + 2y + z = z \end{cases} \Rightarrow \begin{cases} -x + 2y - z = 0 \\ 3x - 3y = 0 \\ -2x + 2y = 0 \end{cases} \Rightarrow x = y = z$$

So, $E_1 = \{x(1, 1, 1) \mid x \in \mathbb{R}\}$, where $v_1 = (1, 1, 1)$ is the eigenvector associated with $\lambda = 1$.
• For $\lambda = 2$

$$E_2 = \{v = (x, y, z) \in \mathbb{R}^3 \mid Av = 2v\} \Rightarrow \begin{cases} 2y - z = 2x \\ 3x - 2y = 2y \\ -2x + 2y + z = 2z \end{cases} \Rightarrow \begin{cases} -2x + 2y - z = 0 \\ 3x - 4y = 0 \\ -2x + 2y - z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{4}{3}y \\ z = \frac{-2}{3}y \end{cases}$$

So, $E_2 = \{y(\frac{4}{3}, 1, \frac{-2}{3}) \mid y \in \mathbb{R}\}$, where $v_2 = (4, 3, -2)$ is the eigenvector associated with $\lambda = 2$.
• For $\lambda = -4$

$$E_{-4} = \{v = (x, y, z) \in \mathbb{R}^3 \mid Av = -4v\} \Rightarrow \begin{cases} 2y - z = -4x \\ 3x - 2y = -4y \\ -2x + 2y + z = -4z \end{cases} \Rightarrow \begin{cases} 4x + 2y - z = 0 \\ 3x + 2y = 0 \\ -2x + 2y + 5z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{-2}{3}y \\ z = x \end{cases}$$

So, $E_{-4} = \{x(1, \frac{-3}{2}, 1) \mid x \in \mathbb{R}\}$, where $v_3 = (2, -3, 2)$ is the eigenvector associated with $\lambda = -4$.
Thus,

$$P = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 3 & -3 \\ 1 & -2 & 2 \end{bmatrix}$$

and $A = PDP^{-1}$, with

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

and

$$P^{-1} = \frac{1}{\det P} C_p = \frac{1}{-30} \begin{pmatrix} 0 & -12 & -18 \\ -5 & 0 & 5 \\ -5 & 6 & -1 \end{pmatrix} \Rightarrow C_p^t = \begin{pmatrix} 0 & -5 & -5 \\ -12 & 0 & 6 \\ -18 & 5 & -1 \end{pmatrix}$$

We compute $P^{-1}$ by using the change of basis.
Then

$$A^k = PD^kP^{-1} = \begin{pmatrix} (-5)2^{k+2} - 10(-4)^k & -12 + 12(-4)^k & -18 + (5)2^{k+2} - 2(-4)^k \\ (-15)2^k - 15(-4)^k & -12 + 15(-4)^k & -18 + (5)2^{k+1} + 3(-4)^k \\ (5)2^{k+1} - 10(-4)^k & -12 + 15(-4)^k & -18 + (5)2^{k+1} - 2(-4)^k \end{pmatrix}$$