Chapter 3

Matrices

3.1 Matrix Associated with a Linear Map

Let $\mathbb{K}$ be a commutative field.
Let E and F be two $\mathbb{K}$ vector spaces of finite dimension n and m, f a linear map from E to F. Let $B = \{e_1, e_2, ..., e_n\}$ be a basis of E, $B^{'} = \{e_1^{'}, e_2^{'}, ..., e_m^{'}\}$ be a basis of F. Since $f(e_1), f(e_2), ..., f(e_n)$ are vectors in F and $\{e_1^{'}, e_2^{'}, ..., e_m^{'}\}$ is a basis of F, then $f(e_1)$, $f(e_2)$, ..., $f(e_n)$ can be written as linear combinations of the vectors in the basis $B^{'} = \{e_1^{'}, e_2^{'}, ..., e_m^{'}\}$. For every $j = 1, ..., n$, we have :

$$f(e_j) = a_{1j}e_1^{'} + a_{2j}e_2^{'} + ... + a_{mj}e_m^{'} = \sum_{j=1}^{n} a_{ij}e_i^{'}, \qquad i = 1, ..., m$$

Then, we have $(f(e_1), f(e_2), ..., f(e_n)) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} e_1^{'} \\ e_2^{'} \\ \vdots \\ e_m^{'} \end{bmatrix}$

and

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called the matrix associated with f relative to the basis B and $B^{'}$. The matrix is denoted by (a_{ij}) where i denotes the row index and j denotes the column index.

Now, let's introduce the concept of matrices and algebraic operations on matrices.

Definition 3.1.1

A matrix in $\mathbb{K}$ of type (n, p) is a rectangular array A of elements from IK with n rows and p columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}.$$

We denote a_{ij} as the element at row number i and column j, and we represent the matrix A by $A = (a_{ij})_{1\leq i\leq n,\ 1\leq j\leq p}$. The set of matrices of type (n, p) is denoted as $\mathcal{M}_{(n,p)}(IK)$.

1. For $n = 1$, we say that A is a row matrix, $A = (a_{11}, a_{12}, ..., a_{1p})$.
2. For $p = 1$, we say that A is a column matrix, $A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$

3. For $n = p$, we say that A is a square matrix of order n, and we denote $A \in \mathcal{M}_n(\mathbb{K})$.

Example 3.1.2

1. $A_1 = \begin{pmatrix} -10 & 2 & 0 \\ 1 & -4 & 9 \\ -5 & -7 & 0 \\ -3 & -1 & 0 \end{pmatrix}$, A_1 is a matrix of type $(4, 3)$.

2. $A_2 = \begin{pmatrix} -4 & -1 & 3 \\ 0 & -2 & -7 \end{pmatrix}$, A_2 is a matrix of type $(2, 3)$.

3. $A_3 = \begin{pmatrix} -1 & -5 \\ 4 & -4 \end{pmatrix}$, A_3 is a square matrix of order 2.

3.2 Vector Space of Matrices with n Rows and m Columns

3.2.1 Matrix Operations

Definition 3.2.1

Let $A = (a_{ij})_{1\le i\le n,\ 1\le j\le p}$ and $B = (b_{ij})_{1\le i\le n,\ 1\le j\le p}$ be two matrices of types (n, p),

1. We say that $A = B$ if $\forall i = 1, ..., n,\ \forall j = 1, ..., p\,;\ a_{ij} = b_{ij}$.
2. The transpose of matrix A is a matrix denoted by A^t defined by :

$$A^t = (a_{ji})_{1\le j\le p,\ 1\le i\le n}.$$

In other words, A^t is the matrix of type (p, n) obtained by replacing the rows with the columns and the columns with the rows, and we have :

$$(A^t)^t = A.$$

Example 3.2.2

1. $A_1 = \begin{pmatrix} 1 & 0 \\ -4 & 6 \\ -3 & -5 \end{pmatrix} \Rightarrow A_1^t = \begin{pmatrix} 1 & -4 & -3 \\ 0 & 6 & -5 \end{pmatrix}$.

2. $A_2 = \begin{pmatrix} 5 & 7 & 1 & 0 & -10 \\ -8 & 0 & 5 & -13 & 5 \\ 7 & 9 & 3 & -2 & 1 \\ -1 & 0 & 5 & 0 & 0 \end{pmatrix} \Rightarrow A_2^t = \begin{pmatrix} 5 & -8 & 7 & -1 \\ 7 & 0 & 9 & 0 \\ 1 & 5 & 3 & 5 \\ 0 & -13 & -2 & 0 \\ -10 & 5 & 1 & 0 \end{pmatrix}$.

3. $A_3 = \begin{pmatrix} 0 & -1 \\ 5 & -8 \end{pmatrix} \Rightarrow A_3^t = \begin{pmatrix} 0 & 5 \\ -1 & -8 \end{pmatrix}$.

Sum of Matrices

Theorem 3.2.3

By equipping the set $\mathcal{M}_{(n,p)}(\mathbb{K})$ with the following operations :

$$(+) : \mathcal{M}_{(n,p)}(\mathbb{K}) \times \mathcal{M}_{(n,p)}(\mathbb{K}) \to \mathcal{M}_{(n,p)}(\mathbb{K})$$

$$\left(\begin{pmatrix} a_{11} & a_{12} & .. & a_{1p} \\ a_{21} & a_{22} & .. & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & .. & a_{np} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & .. & b_{1p} \\ b_{21} & b_{22} & .. & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & .. & b_{np} \end{pmatrix}\right) \mapsto \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & .. & a_{1p}+b_{1p} \\ a_{21}+b_{21} & a_{22}+b_{22} & .. & a_{2p}+b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & .. & a_{np}+b_{np} \end{pmatrix},$$

and

$$(\times): \quad \mathbb{K} \times \mathcal{M}_{(n,p)}(\mathbb{K}) \quad \to \mathcal{M}_{(n,p)}(\mathbb{K}).$$

$$\left(\lambda, \begin{pmatrix} a_{11} & a_{12} & .. & a_{1p} \\ a_{21} & a_{22} & .. & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & .. & a_{np} \end{pmatrix}\right) \mapsto \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & .. & \lambda a_{1p} \\ \lambda a_{21} & \lambda a_{22} & .. & \lambda a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & .. & \lambda a_{np} \end{pmatrix}.$$

Then $(\mathcal{M}_{(n,p)}(\mathbb{K}), +, \cdot)$ is an $\mathbb{K}$-vector space of dimension $n \times p$,

where the additive identity is the zero matrix $\begin{pmatrix} 0 & 0 & .. & 0 \\ 0 & 0 & .. & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & .. & 0 \end{pmatrix}$..

Product of Two Matrices

Definition 3.2.4

Let $A \in \mathcal{M}_{(n,p)}(\mathbb{K})$ and $B \in \mathcal{M}_{(p,m)}(\mathbb{K})$, the product of matrix A by B is defined as a matrix $C = (c_{ij})_{1 \le i \le n,\ 1 \le j \le m} \in \mathcal{M}(n,m)(\mathbb{K})$, with

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + ... + a_{ip}b_{pj}.$$

Note 3.2.5

1. The element c_{ij} of matrix C is calculated by adding the product of the elements in the i-th row of matrix A by the elements in the j-th column of matrix B.
2. The product of two matrices is possible only if the number of columns in matrix A is equal to the number of rows in matrix B.

Example 3.2.6

$$A = \begin{pmatrix} 1 & 0 & 9 \\ 4 & 7 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 5 & 0 & 2 & 1 \\ 0 & 3 & 0 & 0 \end{pmatrix}$$

A is of type $(2,3)$ and B is of type $(3,4)$, so C will be of type $(2,4)$.

$$C = A{\cdot}B = \begin{pmatrix} 1.2+0.5+9.0 & 1.1+0.(0)+9.3 & 1.(3)+0.(2)+9.0 & 1.(1)+0.(1)+9.0 \\ 4.2+7.5+(-4).0 & 4.1+7.(0)+(-4).3 & 4.(3)+7.(2)+(-4).0 & 4.(1)+7.(1)+(-4).0 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 28 & 3 & 1 \\ 43 & -8 & 26 & 11 \end{pmatrix}$$

Note 3.2.7

The product of two matrices is not commutative, here is an example :

$$A \times B = \begin{pmatrix} 1 & 2 \\ -4 & 5 \end{pmatrix} \times \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 10 & -17 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} -12 & 15 \\ 6 & -1 \end{pmatrix}$$

So $A \times B \neq B \times A$

3.3 Square Matrix Ring

Definition 3.3.1

Let A be a square matrix of order n, $A = (a_{ij})_{1 \leq i \leq n,\ 1 \leq j \leq n}$,

1. The sequence of elements $\{a_{11}, a_{22}, ..., a_{nn}\}$ is called the principal diagonal of A.
2. The trace of A is the number
$$Tr(A) = a_{11} + a_{22} + ... + a_{nn}.$$
3. A is called a diagonal matrix if $a_{ij} = 0$, $\forall i \neq j$, meaning all elements of A are zero except on the principal diagonal.
4. A is called an upper (resp. lower) triangular matrix if $a_{ij} = 0$, $\forall i > j$ (resp. $i < j$), meaning elements below (resp. above) the diagonal are zero.
5. A is called symmetric if $A = A^t$.

Example 3.3.2

1. $A_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, A_1 is a diagonal matrix.

2. $A_2 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -14 & 0 \\ -3 & 7 & 2 \end{pmatrix}$, A_2 is a lower triangular matrix.

3. $A_3 = \begin{pmatrix} 2 & 10 & -25 \\ 0 & -1 & 22 \\ 0 & 0 & 1 \end{pmatrix}$, A_3 is an upper triangular matrix.

4. $A_4 = \begin{pmatrix} 2 & 10 & -2 \\ 10 & -1 & 21 \\ -2 & 21 & 1 \end{pmatrix} \Rightarrow A_4^t = \begin{pmatrix} 2 & 10 & -2 \\ 10 & -1 & 21 \\ -2 & 21 & 1 \end{pmatrix}$, A_4 is a symmetric matrix.

Proposition 3.3.3

The matrix product is an internal operation in $\mathcal{M}_{(n,n)}(IK)$, and it has a neutral element called the identity matrix, denoted by I_n, defined as :

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

3.3.1 Invertible Matrices

Definition 3.3.4

Let $A \in \mathcal{M}_{(n,n)}(IK)$, we say that A is invertible if there exists a matrix $B \in \mathcal{M}_{(n,n)}(IK)$ such that :

$$A.B = B.A = I_n.$$

Example 3.3.5

Show that the matrix $A = \begin{pmatrix} 1 & 1 \\ 4 & -3 \end{pmatrix}$ is invertible.

By seeking the matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that :

$$A.B = \begin{pmatrix} 1 & 1 \\ 4 & -3 \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 4 & -3 \end{pmatrix} = B.A$$

This gives :

$$A.B = \begin{pmatrix} a+c & b+d \\ 4a-3c & 4b-3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+4b & a-3b \\ c+4d & c-3d \end{pmatrix} = B.A$$

Which leads to the system :

$$\Rightarrow \begin{cases} a+c=1 \\ 4a-3c=0 \\ b+d=0 \\ 4b-3d=1 \end{cases} \Rightarrow \begin{cases} a=\frac{3}{7} \\ c=\frac{4}{7} \\ b=\frac{1}{7} \\ d=\frac{-1}{7} \end{cases}$$

So, $B = \begin{pmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{pmatrix}$. Then, $B = \frac{1}{7}.\begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}$

3.3.2 Determinant of a Square Matrix and Properties

Definition 3.3.6

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, be a matrix in $\mathcal{M}_{(2,2)}(\mathbb{R})$, the determinant of A is the real number given by :

$$det(A) = a_{11} \times a_{22} - a_{12}a_{21}$$

It is denoted as $det(A)$ or $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

Example 3.3.7

Calculate $det(A)$ for $A = \begin{pmatrix} 1 & -4 \\ 2 & 5 \end{pmatrix}$.

$$det(A) = |A| = \begin{vmatrix} 1 & -4 \\ 2 & 5 \end{vmatrix} = 1 \times 5 - 2 \times (-4) = 13$$

Definition 3.3.8

Similarly, the determinant of a 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathcal{M}_{(3,3)}(\mathbb{R})$ is given by :

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Where C_{11}, C_{12}, and C_{13} are the cofactors of the elements a_{11}, a_{12}, and a_{13}, respectively.

Example 3.3.9

Calculate $|A|$ for $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 2 & 1 & -1 \end{pmatrix}$.

$$|A| = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= 1(-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} + 0(-1)^{1+3}\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= 1(-3-1) - 2(1-2) + 0(-1-6) = -2.$$

Proposition 3.3.10

To calculate the determinant of a matrix A, one can expand A along any row or column. Following this proposition, it is better to choose the row or column containing the most zeros.

Example 3.3.11

Using the same matrix as the previous example.
Method 1 : calculating along the third row, we have :

$$det(A) = |A| = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 2 & 1 & -1 \end{vmatrix} = (-1)^{3+1}(2)\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + (-1)^{3+2}(+1)\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + (-1)^{3+3}(-1)\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

Therefore :

$$det(A) = |A| = 2(2) - 1 - 5 = 4 - 6 = -2.$$

Method 2 : Calculate $|A|$ for

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

.

$$det(A) = |A| = a_{11}(a_{22})(a_{23}) + a_{12}(a_{23})(a_{31}) + a_{21}(a_{32})(a_{13}) - [a_{31}(a_{22})(a_{13}) + a_{32}(a_{23})(a_{11}) + a_{21}(a_{12})(a_{33})]$$

$$= 1(3)(-1) + 2(1)(2) + 1(-1)(0) - [2(3)(0) + 1(1)(1) + 2(-1)(-1)] = 1 - 3 = -2$$

Definition 3.3.12

Similarly, the determinant of a 4×4 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \mathcal{M}_{(4,4)}(\mathbb{R})$ is given by :

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = (-1)^{1+1}(a_{11}) \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} + \ldots + (-1)^{1+4}(a_{14}) \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Definition 3.3.13

Let $A = (a_{ij})_{1 \le i \le n, 1 \le j \le n}$ be a matrix. The determinant along the j-th column is given by :

$$\det(A) = (-1)^{1+j} a_{1j} D_{1j} + (-1)^{2+j} a_{2j} D_{2j} + \cdots + (-1)^{n+j} a_{nj} D_{nj}, \quad j = 1, \ldots, n.$$

The determinant along the i-th row is given by :

$$\det(A) = (-1)^{i+1} a_{i1} D_{i1} + (-1)^{i+2} a_{i2} D_{i2} + \cdots + (-1)^{i+n} a_{in} D_{in}, \quad i = 1, \ldots, n.$$

Here, A_{ij} represents the minor determinant of the term a_{ij}, which is the determinant of order $n-1$ obtained from $\det(A)$ by removing the i-th row and j-th column.

Proposition 3.3.14

Let $A \in \mathcal{M}_n(IK)$. We have :

1. $\det(A) = \det(A^t)$.
2. $\det(A) = 0$ if two rows (or two columns) of A are equal.
3. $\det(A) = 0$ if two rows (or two columns) of A are proportional.
4. $\det(A) = 0$ if one row is a linear combination of two other rows of A (similarly for columns).
5. $\det(A)$ remains unchanged if a linear combination of other rows is added to one row (similarly for columns).
6. If $B \in \mathcal{M}_n(IK)$, then $\det(A \cdot B) = \det(A) \cdot \det(B)$.

Example 3.3.15

1. $|A| = \begin{vmatrix} 1 & 0 & -5 \\ 5 & 4 & -2 \\ 1 & 0 & -5 \end{vmatrix} = 0$, because row 1 is equal to row 3, $L_1 = L_3$.

2. $|B| = \begin{vmatrix} 2 & -4 & 6 & 10 \\ 0 & 8 & -7 & 1 \\ 1 & -2 & 3 & 5 \\ 2 & -3 & 0 & -1 \end{vmatrix} = 0$, because $L_1 = 2L_3$.

3. $|C| = \begin{vmatrix} 2 & -3 & 2 & 6 \\ 1 & 8 & 1 & -1 \\ 0 & -4 & 0 & 5 \\ -2 & -3 & -2 & -1 \end{vmatrix} = 0$, because $C_1 = C_3$.

Definition 3.3.16

Let $V_1, V_2, \ldots, V_n$ be n vectors in $\mathbb{R}^n$. The determinant of the vectors $(V_1, V_2, \ldots, V_n)$, denoted as

$$\det(V_1, V_2, \ldots, V_n)$$

, is the determinant whose columns are the vectors $V_1, V_2, \ldots, V_n$.

Example 3.3.17

Let $V_1 = (-1, -1, 0)$, $V_2 = (0, -1, 0)$, $V_3 = (0, 1, 1)$. Then

$$\begin{aligned}\det(V_1, V_2, V_3) &= \begin{vmatrix} -1 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \\ &= (-1)\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} \\ &= (-1)(-1) \\ &= 1.\end{aligned}$$

Proposition 3.3.18

The vectors $(V_1, V_2, \ldots, V_n)$ form a basis for $\mathbb{R}^n$ if and only if $\det(V_1, V_2, \ldots, V_n) \neq 0$.

Example 3.3.19

Let $V_1 = (-1, -2, 0)$, $V_2 = (0, -1, -1)$, $V_3 = (0, 2, 1)$. They form a basis for $\mathbb{R}^3$ because $\det(V_1, V_2, V_3) \neq 0$.

3.3.3 Rank of a Matrix (Associated Application)

Definition 3.3.20

Let $A \in \mathcal{M}(n, p)(IK)$. The rank of A, denoted as rgA, is the order of the largest square matrix B extracted from A such that $\det(B) \neq 0$.

Example 3.3.21

1. $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \Rightarrow \det(A) = 1 \times 4 - (-1) \times 2 = 6 \neq 0 \Rightarrow rgA = 2.$

2. $B = \begin{pmatrix} 1 & -4 \\ -1 & 4 \end{pmatrix} \Rightarrow \det(B) = 0\,;$ thus $rgB = 1.$

3. $C = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & -1 \end{pmatrix}$, $rgC < 4$ $(rgC \leq 3)$. The largest square matrix contained in C is of

order 3. In this example, there are 4 possibilities :

$$C_1 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

$\det(C_1) = \det(C_2) = \det(C_3) = \det(C_4) = 0$, so $rgC < 3$, and we have :

$$\begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow rgC = 2.$$

Theorem 3.3.22

The rank of a matrix is equal to the maximum number of linearly independent row (or column) vectors.

Definition 3.3.23

Let $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq n} \in \mathcal{M}_n(IK)$. The cofactor of index i and j of A is the scalar $c_{ij} = (-1)^{i+j}\det A_{i_j}$, where A_{ij} is the matrix obtained from A by removing the i-th row and the j-th column. The matrix $C = (c_{ij})_{1\leq i\leq n, 1\leq j\leq n}$ is called the matrix of cofactors, and the transpose of C is called the adjugate or comatrix of A.

Example 3.3.24

Consider the matrix $A = \begin{pmatrix} 1 & -2 & 0 \\ 5 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.

The cofactors of A are calculated as follows :

$$c_{11} = (-1)^{1+1}\det(A_{11}) = (-1)^2 \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -4,$$

$$c_{12} = (-1)^{1+2}\det(A_{12}) = (-1)^3 \begin{vmatrix} 5 & 1 \\ 0 & -1 \end{vmatrix} = 5,$$

$$c_{13} = (-1)^{1+3}\det(A_{13}) = (-1)^4 \begin{vmatrix} 5 & 3 \\ 0 & 1 \end{vmatrix} = 5,$$

$$c_{21} = (-1)^{2+1}\det(A_{21}) = (-1)^3 \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} = -2,$$

$$c_{22} = (-1)^{2+2}\det(A_{22}) = (-1)^4 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1,$$

$$c_{23} = (-1)^{2+3}\det(A_{23}) = (-1)^5 \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1,$$

$$c_{31} = (-1)^{3+1}\det(A_{31}) = (-1)^4 \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix} = -2,$$

$$c_{32} = (-1)^{3+2}\det(A_{32}) = (-1)^5 \begin{vmatrix} 1 & 0 \\ 5 & 1 \end{vmatrix} = -1,$$

$$c_{33} = (-1)^{3+3}\det(A_{33}) = (-1)^6 \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix} = 13.$$

The matrix of cofactors is :

$$C = \begin{pmatrix} -4 & 5 & 5 \\ -2 & -1 & 1 \\ -2 & -1 & 13 \end{pmatrix}$$

The transpose of the matrix of cofactors (adjugate or comatrix of A) is :

$$C^{\,t} = \begin{pmatrix} -4 & -2 & -2 \\ 5 & -1 & -1 \\ 5 & 1 & 13 \end{pmatrix}$$

Theorem 3.3.25

Let $A \in \mathcal{M}_n(IK)$, then :

$$A \text{ is invertible} \Leftrightarrow \det(A) \neq 0$$

In this case, the inverse of matrix A is given by :

$$A^{-1} = \frac{1}{\det(A)} C^{\,t}$$

where $C^{\,t}$ is the adjugate (or comatrix) of A.

Example 3.3.26

Consider the matrix $A = \begin{pmatrix} 1 & -2 & 0 \\ 5 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.

Here, $\det(A) = -14 \neq 0$, so A is invertible. The inverse of A is calculated as :

$$A^{-1} = \frac{1}{\det(A)} C^{\,t} = \frac{1}{-14} \begin{pmatrix} -4 & -2 & -2 \\ 5 & -1 & -1 \\ 5 & 1 & 13 \end{pmatrix} = \begin{pmatrix} \frac{4}{14} & \frac{2}{14} & \frac{2}{14} \\ \frac{-5}{14} & \frac{1}{14} & \frac{1}{14} \\ \frac{-5}{14} & \frac{1}{14} & \frac{-13}{14} \end{pmatrix}$$

One can verify that :

$$A^{-1}A = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = AA^{-1}.$$

3.4 Relations between a linear map and its associated matrix

Definition 3.4.1

Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be the matrix of f with respect to the basis B and $B^{'}$. If $E = F$ and $B = B^{'}$, we say that A is the matrix of f with respect to the base B and denote it by $\mathcal{M}_{(B)}(f)$.

Example 3.4.2

1. Consider the linear map :

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x, y, z) \mapsto f(x, y, z) = (x + y + z, x - y).$$

Using the canonical basis $B = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ for $\mathbb{R}^3$ and $B^{'} = \{v_1 = (1, 0), v_2 = (0, 1)\}$ for $\mathbb{R}^2$, we have :

$$\begin{aligned} f(x, y, z) &= f(xe_1 + ye_2 + ze_3) \\ &= f(xe_1) + f(ye_2) + f(ze_3) \\ &= f(x(1, 0, 0)) + f(y(0, 1, 0)) + f(z(0, 0, 1)) \\ &= f((x, 0, 0)) + f((0, y, 0)) + f((0, 0, z)) \\ &= (x, x) + (y, -y) + (z, 0) \\ &= x(1, 1) + y(1, -1) + z(1, 0) \end{aligned}$$

So,

$$\mathcal{M}_{(B,B')}(f) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

2. Consider the linear map :

$$g : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto f(x, y) = (x + y, x - y).$$

With basis $B = \{e_1 = (1, 2), e_2 = (-1, 1)\}$ and $B^{'} = \{v_1 = (0, 2), v_2 = (-2, 1)\}$.

$$\begin{aligned} f(1, 2) = (3, -1) &= \lambda_1 v_1 + \lambda_2 v_2 \\ &= \lambda_1(0, 2) + \lambda_2(-2, 1) \\ &= (0, 2\lambda_1) + (-2\lambda_2, \lambda_2) \\ &= (-2\lambda_2, 2\lambda_1 + \lambda_2) \end{aligned}$$

$-2\lambda_2 = 3 \Rightarrow \lambda_2 = \frac{-3}{2}$
So,
$2\lambda_1 + \lambda_2 = 2\lambda_1 + \frac{-3}{2} = -1 \Rightarrow \lambda_1 = \frac{1}{4}$.
Then $(\lambda_1, \lambda_2) = (\frac{1}{4}, \frac{-3}{2})$.
Now,

$$\begin{aligned} f(-1, 1) = (0, -2) &= \lambda_1 v_1 + \lambda_2 v_2 \\ &= \lambda_1(0, 2) + \lambda_2(-2, 1) \\ &= (0, 2\lambda_1) + (-2\lambda_2, \lambda_2) \\ &= (-2\lambda_2, 2\lambda_1 + \lambda_2) \end{aligned}$$

$-2\lambda_2 = 0 \Rightarrow \lambda_2 = 0$
So,
$2\lambda_1 + \lambda_2 = 2\lambda_1 = -2 \Rightarrow \lambda_1 = -1$.
Then $(\lambda_1, \lambda_2) = (-1, 0)$.
We find :

$$\mathcal{M}_{(B,B')}(g) = \begin{pmatrix} \frac{1}{4} & -1 \\ \frac{-3}{2} & 0 \end{pmatrix}$$

Proposition 3.4.3

Let E and F be two $\mathbb{K}$-vector spaces of dimensions n and m, and $B = (e_1, e_2, ..., e_n)$ a basis of E, and $B^{'} = (v_1, v_2, ..., v_m)$ a basis of F. Then, the matrix $A \in \mathcal{M}_{(n,m)}(\mathbb{K})$ corresponds to a unique linear map f from E to F. The matrix representation of f with respect to basis B and $B^{'}$ is given by A.

Example 3.4.4

Consider the matrix $A = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}$. We have :

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+4y \\ 2x-3y \end{pmatrix}.$$

The associated linear application is :

$$\begin{aligned} &f : \mathbb{R}^2 \to \mathbb{R}^2, \\ &(x,y) \mapsto f(x,y) = (-x+4y, 2x-3y). \end{aligned}$$

Note 3.4.5

If $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with their canonical bases, the linear map f from $\mathbb{R}^n$ to $\mathbb{R}^m$ associated with a matrix $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ is given by :

$$f(x_1, x_2, ..., x_n) = A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Example 3.4.6

Consider the matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix}$. We have :

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x-y \\ y-2z \\ -x+z \end{pmatrix}.$$

The associated linear map is :

$$\begin{aligned} &f : \mathbb{R}^3 \to \mathbb{R}^3, \\ &(x,y,z) \mapsto f(x,y,z) = (2x-y, y-2z, -x+z). \end{aligned}$$

Theorem 3.4.7

Let E, F, and G be IK-vector spaces with basis B, $B^{'}$, and $B^{''}$, respectively.
If $f : E \to F$ and $g : F \to G$ are linear maps, then :

$$\mathcal{M}(B, B^{''})(g \circ f) = \mathcal{M}(B^{'}, B^{''})(g) \cdot \mathcal{M}(B, B^{'})(f).$$

Example 3.4.8

Consider linear applications :

$$\begin{aligned} &f : \mathbb{R}^3 \to \mathbb{R}^2, \\ &(x,y,z) \mapsto f(x,y,z) = (2x+y-z, 2x-y). \end{aligned}$$

And

$$\begin{aligned} &g : \mathbb{R}^2 \to \mathbb{R}^2, \\ &(x,y) \mapsto g(x,y) = (x-2y, x+y). \end{aligned}$$

The matrix representations are :

$$\mathcal{M}(B, B^{''})(g \circ f) = \begin{pmatrix} -2 & 3 & -1 \\ 4 & 0 & -1 \end{pmatrix}$$

$$\mathcal{M}(B^{'}, B^{''})(g) = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$$

$$\mathcal{M}(B, B^{'}) = \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\mathcal{M}(B^{'}, B^{''})(g) \cdot \mathcal{M}(B, B^{'})(f) = \mathcal{M}(B, B^{''})(g \circ f)$$

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 3 & -1 \\ 4 & 0 & -1 \end{pmatrix}$$

Theorem 3.4.9

For linear application $f : E \to F$, where B is a basis of E and $B^{'}$ is a basis of F :

$$f \text{ is bijective} \Leftrightarrow \det\Big(\mathcal{M}_{(B,B')}(f)\Big) \neq 0.$$

In this case :

$$\mathcal{M}_{(B,B')}(f^{-1}) = (\mathcal{M}_{(B,B')}(f))^{-1}.$$

Example 3.4.10

Consider the linear application :

$$f : \mathbb{R}^2 \to \mathbb{R}^2,$$
$$(x, y) \mapsto f(x, y) = (-x + y, x + y).$$

Show that f is bijective and calculate its inverse.
The matrix representation is :

$$\mathcal{M}_2(f) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The inverse is given by :

$$(\mathcal{M}_2(f^{-1}))^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Therefore, $f^{-1}(x, y) = \left(\frac{1}{2} - \frac{1}{2}y, -\frac{1}{2}x - \frac{1}{2}y\right)$.

3.5 Invariance of rank under transposition

Proposition 3.5.1

If $A \in \mathcal{M}_{(n,m)}(\mathbb{K})$ is associated with a linear map $f : E \to F$, where B is a basis of E and $B^{'}$ is a basis of F, then :

$$\text{rank}(A) = \text{rank}(f).$$

Also,

$$\text{rank}(A) = \text{rank}(A^{\,t})$$

3.6 Matrices and Change of Basis

Definition 3.6.1

Let E be a vector space, and let $B = (e_1, e_2, \ldots, e_n)$ and $B' = (e'_1, e'_2, \ldots, e'_n)$ be two basis for E. The change of basis matrix from B' to B is, by definition, the matrix $\det\big(\mathcal{M}_{(B,B')}(Id_E)\big)$, where Id_E is the identity map :

$$\begin{aligned} Id_E : E &\to E \\ x &\mapsto Id'_E(x) = x. \end{aligned}$$

The basis vectors of B can be expressed in B' according to the relations :

$$S : \begin{cases} e_1 = a_{11}e'_1 + a_{12}e'_2 + a_{13}e'_3 + \cdots + a_{1n}e'_n \\ e_2 = a_{21}e'_1 + a_{22}e'_2 + a_{23}e'_3 + \cdots + a_{2n}e'_n \\ e_3 = a_{31}e'_1 + a_{32}e'_2 + a_{33}e'_3 + \cdots + a_{3n}e'_n \\ \vdots \\ e_n = a_{n1}e'_1 + a_{n2}e'_2 + a_{n3}e'_3 + \cdots + a_{nn}e'_n. \end{cases}$$

The matrix of passage from B' to B is the square matrix P defined by :

$$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Example 3.6.2

$B' = (e'_1, e'_2, e'_3)$, $e'_1 = (1,1,1)$, $e'_2 = (1,1,0)$, $e'_3 = (1,0,0)$ and $B = (e_1, e_2, e_3)$ is the canonical basis of $\mathbb{R}^3$.

$$\begin{aligned} Id_E : \mathbb{R}^3_B &\to \mathbb{R}^3_{B'} \\ (x,y,z) &\mapsto Id_{\mathbb{R}^3}(x,y,z) = (x,y,z) \end{aligned}$$

$\mathcal{M}_{(B,B')}(Id_{\mathbb{R}^3}) = ?$

$$S : \begin{cases} Id\ (e_1) = (1,0,0) = a_{11}e'_1 + a_{12}e'_2 + a_{13}e'_3 \\ Id\ (e_2) = (0,1,0) = a_{21}e'_1 + a_{22}e'_2 + a_{23}e'_3 \\ Id\ (e_3) = (0,0,1) = a_{31}e'_1 + a_{32}e'_2 + a_{33}e'_3 \end{cases}$$

$$S \Leftrightarrow \begin{cases} Id\ (e_1) = (1,0,0) = a_{11}(1,1,1) + a_{12}(1,1,0) + a_{13}(1,0,0) \\ Id\ (e_2) = (0,1,0) = a_{21}(1,1,1) + a_{22}(1,1,0) + a_{23}(1,0,0) \\ Id\ (e_3) = (0,0,1) = a_{31}(1,1,1) + a_{32}(1,1,0) + a_{33}(1,0,0) \end{cases}$$

$$S \Leftrightarrow \begin{cases} a_{11} = \frac{1}{2}, \quad a_{12} = -\frac{1}{2}, \quad a_{13} = \frac{1}{2} \\ a_{21} = \frac{1}{2}, \quad a_{22} = \frac{1}{2}, \quad a_{23} = -\frac{1}{2} \\ a_{31} = -\frac{1}{2}, \quad a_{32} = \frac{1}{2}, \quad a_{33} = \frac{1}{2} \end{cases}$$

Therefore :

$$\mathcal{M}_{(B,B')}(Id_{\mathbb{R}^3}) = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

Proposition 3.6.3

The change of basis matrix from a basis B to a basis B' is the inverse matrix of the change of basis matrix from B' to B :

$$\mathcal{M}_{(B',B)}(Id_{\mathbb{R}^3}) = \left(\mathcal{M}_{(B,B')}(Id_{\mathbb{R}^3})\right)^{-1}.$$

Note 3.6.4

Let E be a vector space, and let $B = (e_1, e_2, \ldots, e_n)$ and $B' = (e'_1, e'_2, \ldots, e'_n)$ be two basis for E. The basis vectors of B' can be expressed in B according to the relations :

$$S : \begin{cases} e'_1 = b_{11}e_1 + b_{12}e_2 + a_{13}e_3 + \cdots + b_{1n}e_n \\ e'_2 = b_{21}e_1 + b_{22}e_2 + b_{23}e_3 + \cdots + b_{2n}e_n \\ e'_3 = b_{31}e_1 + b_{32}e_2 + b_{33}e_3 + \cdots + b_{3n}e_n \\ \vdots \\ e'_n = b_{n1}e_1 + b_{n2}e_2 + b_{n3}e_3 + \cdots + b_{nn}e_n \end{cases}$$

The change of basis matrix from B to B' is the square matrix P^{-1} defined by :

$$P^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix}$$

Example 3.6.5

$$\mathcal{M}_{(B,B')}(Id_{\mathbb{R}^3}) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\mathcal{M}_{(B',B)}(Id_{\mathbb{R}^3}) = \left[\frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}\right]^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Change of basis matrix from basis B to B'.

Theorem 3.6.6

Let $f : E \to F$, B_1, B'_1 be two basis for E, B_2, B'_2 bases of F.
If P denotes the change of basis matrix from B_1 to B'_1, and Q denotes the change of basis matrix from B_2 to B'_2, then

$$\mathcal{M}_{(B'_1,B'_2)}(f) = Q^{-1}\mathcal{M}_{(B_1,B_2)}(f)P.$$

Example 3.6.7

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$(x, y, z) \mapsto (x + y + z, x - y)$$

We equip $\mathbb{R}^3$ with the canonical basis $B_3 = (e_1, e_2, e_3)$ and $\mathbb{R}^2$ with the canonical basis $B_2 = (v_1, v_2)$. The matrix representation $\mathcal{M}(B_3, B_2)(f)$ is given by

$$\mathcal{M}(B_3, B_2)(f) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

We also consider another basis $B'_3 = (e'_1, e'_2, e'_3)$ for $\mathbb{R}^3$ with $e'_1 = (1, 0, 1)$, $e'_2 = (1, 1, 0)$, $e'_3 = (0, 1, 1)$. For $\mathbb{R}^2$, we have the basis $B'_2 = (v'_1, v'_2)$ with $v'_1 = (-1, 1)$ and $v'_2 = (1, 1)$.

The change of basis matrices are given by

$$P = \mathcal{M}_{(B_3', B_3)}(\mathrm{Id}_{\mathbb{R}^3}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$Q = \mathcal{M}_{(B_2', B_2)}(\mathrm{Id}_{\mathbb{R}^2}) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

The linear transformation f can be represented as

$$\mathcal{M}_{(B_2', B_3')}(f) = Q^{-1}_{(B_2, B_2')} P = Q^{-1} M_{(B_3, B_2')}(f) P$$

Substituting the matrices, we get

$$\mathcal{M}_{(B_2', B_3')}(f) = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & -1 & -1/2 \\ 3/2 & 1 & 1/2 \end{bmatrix}$$

3.7 Diagonalization

Definition 3.7.1

Let $A \in M(n, n)(\mathbb{K})$ and $\lambda \in \mathbb{K}$. We say that λ is an eigenvalue of A if there exists a non-zero column vector v such that $Av = \lambda v$. The vector v is called the eigenvector associated with the eigenvalue λ.

Example 3.7.2

Consider the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$.
For $\lambda_1 = 1$, the corresponding eigenvector is

$$v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

and for $\lambda_2 = 2$, the corresponding eigenvector is

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Proposition 3.7.3

Let $A \in M(n, n)(\mathbb{K})$ and $\lambda \in \mathbb{K}$. λ is an eigenvalue of A if and only if $P_A(\lambda) = \det(A - \lambda I_n) = 0$. Here, $P_A(\lambda)$ is called the characteristic polynomial of A.

Example 3.7.4

For the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

The characteristic polynomial is

$$P_A(\lambda) = \det(A - \lambda I_2) = (2 - \lambda)(1 - \lambda) = 0$$

Then

$$P_A(\lambda) = \det\left(\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$
$$= \det\left(\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$
$$= \det\left(\begin{pmatrix} 2-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix}\right)$$
$$= (2-\lambda)(1-\lambda) = 0$$

This gives the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$.

Definition 3.7.5

Let $A \in M(n,n)(\mathbb{K})$ and $\lambda \in \mathbb{K}$. If λ is an eigenvalue of A, then the set

$$E = \{v \in \mathbb{R}^n \text{ or } \mathbb{C}^n : Av = \lambda v\}$$

is called the eigenspace associated with the eigenvalue λ. The set E is a subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$.

Example 3.7.6

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

The eigenvalues of A are 2 (double) and 1 (simple). For $\lambda = 2$, the eigenspace E_2 is spanned by the vectors $(1,1,0)$ and $(0,1,1)$. For $\lambda = 1$, the eigenspace E_1 is spanned by the vector $(1,1,0)$.

Definition 3.7.7

A matrix $A \in \mathcal{M}_n(IK)$ is said to be diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ (where P is the change of basis matrix).

Theorem 3.7.8

Let $A \in \mathcal{M}_n(IK)$, and let $\lambda_1, \lambda_2, ..., \lambda_n \in IK$ be the eigenvalues of A with respective multiplicities $m_1, m_2, ..., m_p$. Then, if either :

1. $dimE_{\lambda_i} = m_i$ for $i = 1, 2, ..., p$. or
2. $dimE_{\lambda_1} + dimE_{\lambda_2} + ... + dimE_{\lambda_p} = n$.

Then, matrix A is diagonalizable, and the associated diagonal matrix D is given by :

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \lambda_p & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \lambda_n \end{bmatrix}$$

Each entry repeats m_i times, and matrix P is formed by the eigenvectors.

Note 3.7.9

If matrix $A \in \mathcal{M}_n(IK)$ has n distinct eigenvalues, then A is diagonalizable, and the associated diagonal matrix D is :

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

Example 3.7.10

Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. It has eigenvalues $\lambda_1 = 2$ (double) and $\lambda_2 = 1$ (simple). The diagonal matrix D is :

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

And the change of basis matrix P is :

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

3.8 Exercises

Exercise 3.8.1

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

1. Calculate A^2 and A^3. Evaluate $A^3 - A^2 + A - I$.
2. Express A^{-1} in terms of A^2, A, and I.
3. Express A^4 in terms of A^2, A, and I.

Exercise 3.8.2

Let $A = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{bmatrix}$

Calculate $(A - 2I)^3$, then deduce that A is invertible and find A^{-1} in terms of I, A, and A^2.

Exercise 3.8.3

Consider the matrix A defined as :

$$A = \begin{bmatrix} 5 & 6 & -3 \\ -18 & -19 & 9 \\ -30 & -30 & 14 \end{bmatrix}$$

1. Is A invertible ? If yes, determine its inverse A^{-1}.
2. Calculate $A^2 - A - 2I_3 = 0$, where I_3 is the identity matrix.