1. Introduction

A member subjected to forces or torques that lie in a plane containing the longitudinal axis of the member is called a ***beam***. The forces are understood to act perpendicular to the longitudinal axis.

Cantilever Beams*:* If a beam is supported at only one end and in such a manner that the axis of the beam cannot rotate at that point, it is called a cantilever beam (Figure 1). The reaction of the supporting wall upon the beam consists of a ***vertical force*** together with a ***couple*** acting in the plane of the applied loads shown.

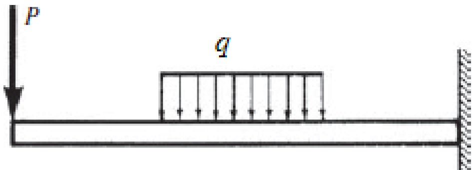


Figure 1 : Poutre Cantilever

Simple Beams: A beam that is freely supported at both ends is called a simple beam (Figure 2a et b). The end supports are capable of exerting only forces upon the bar and are not capable of exerting any moments.

Figure 2 : Simple beam

Overhanging Beams: A beam freely supported at two points and having one or both ends extending beyond these supports is termed an overhanging beam. Twoexamples are given in Fig. 3.

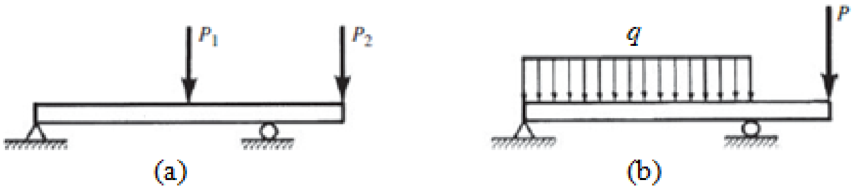


Figure 3 : Poutres en porte – à – faux

A beam is subject to ***bending*** when the forces applied to it tend to vary its curvature. Bending is said to be ***simple*** when the external forces act in the beam's plane of symmetry perpendicular to its axis.

Fundamental assumptions

a - *The deformations are elastic and sufficiently small;*

b- *Any fiber contained within a plane of symmetry remains within that plane during deformation ;*

c- ***The cross-sections of the beam remain flat and perpendicular to its axis after deformation (Navier-Bernoulli Hypothesis).***

2. Internalbending forces

Simple plane bending generates two internal force factors in the cross-sections of a beam: the **bending moment** M and the **shear force** Q.

To determine these, we apply the method of sections. At the location of interest, let's mentally make the cut at distance x from the left support. Consider, for example, the right-hand side, the equilibrium of the left-hand side illustrated in Figure 4.

Figure 4

The internal forces $\boldsymbol{M_z}$ and $\boldsymbol{Q_y}$will be determined from the equilibrium equations of the part considered.

$$\sum F_x = 0 \quad \Rightarrow \quad Q_y = -P_1 + R_A$$

Generalize: $Q_y = \sum F_{iy}$ (1)

Let us calculate the sum of the moments of the forces about the cutting point $\boldsymbol{c}$:

$$\sum M_c = 0 \quad \Rightarrow \quad M_z = R_A . x - P_1 . (x - a)$$

Likewise, to generalize :

$$M_z = \sum M_c(F_i) \qquad (2)$$

Note : before any study of internal efforts, we will first determine the reactions to the supports.

2.1 Sign convention for internal forces

The figures below represent the sign convention used for the internal forces $\boldsymbol{M_z}$ and $\boldsymbol{Q_y}$.

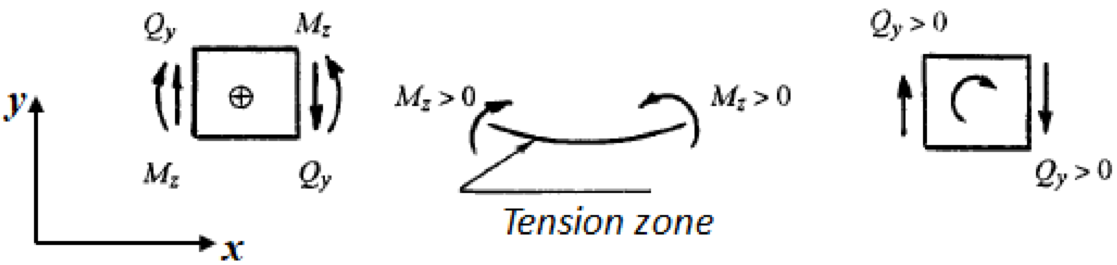


Figure 5

2.2 Diffrential equation for beam bending

The shear force $\boldsymbol{Q_y}$ and the bending moment $\boldsymbol{M_z}$ are related by differential equations, as well for the distributed load $q(x)$, which will be demonstrated later. Consider the equilibrium of element dx in Figure 6:

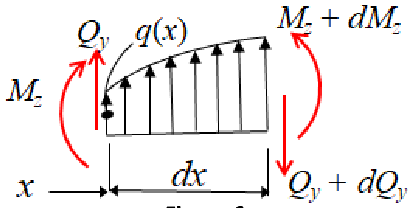


Figure 6

It is easily demonstrated that:

$$\frac{\boldsymbol{dQ_y}}{\boldsymbol{dx}} = \boldsymbol{q(x)} \qquad (3)$$

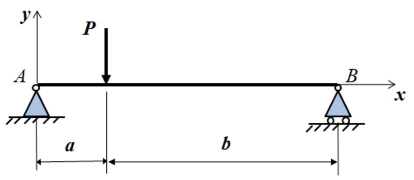
$$\frac{\boldsymbol{dM_z}}{\boldsymbol{dx}} = \boldsymbol{Q_y} \qquad (4)$$

3. Shear force and bending moment diagrams

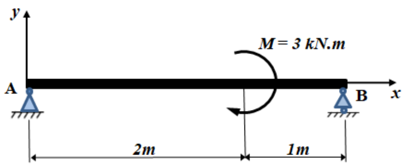
The variation of the shear force and bending moment along a beam subjected to a given load is expressed by plotting diagrams representing the shape of the functions $Q_y(x)$ and $M_z(x)$.

The method of sections will be specifically applied. In practice, along the length of the beam and as a function of $\boldsymbol{x}$, as many sections as there are forces will be made. Relationships 3 and 4 will allow us to analyze the results obtained. In the following, we consider three typical examples, through which we will explain the method in more detail.

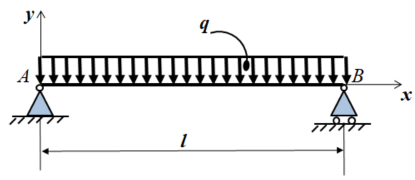
For the beams in the figures below (examples 1, 2 and 3), draw the diagrams of the shear force and the bending moment.



(1)



(2)



(2)

Figure 7

4. Normal and tangential stresses in bending

Simple bending generates two types of stresses:

- Normal stresses, due to the bending moment M_z;

- Tangential stresses, due to the shear force Q_y.

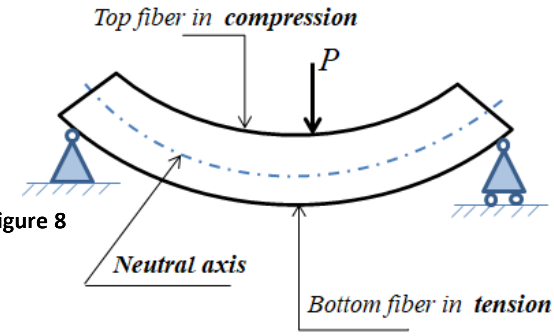


Figure 8

4.1 Study of bending deformations

Consider an element dS of the beam before and after deformation:

Before deformation **After deformation**

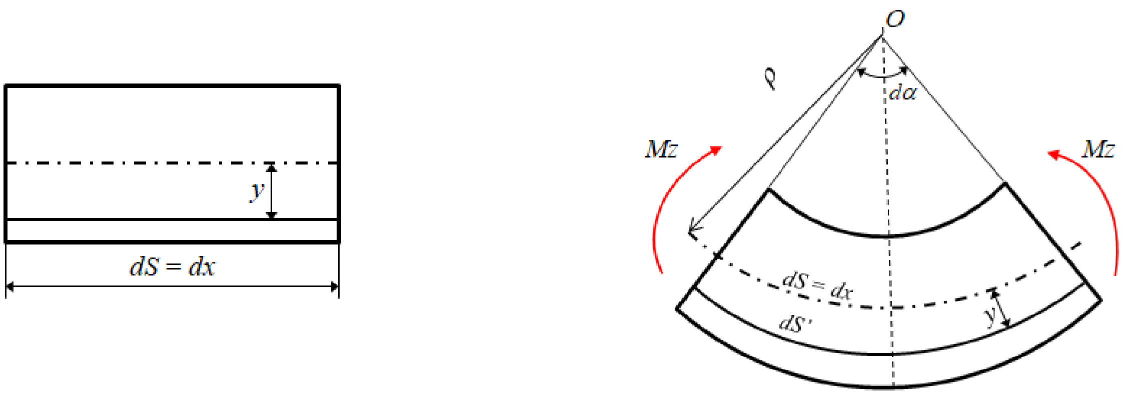


Figure 9

Let ρ be the radius of curvature of the element dS of the deformed shape. The relative deformation in the direction of the axis is given by:

$$\varepsilon_x = \frac{dS' - dS}{dS} = \frac{(\rho + y)d\alpha - \rho d\alpha}{\rho d\alpha} = \frac{y}{\rho}$$

The normal strain along the cross section is:

$$\varepsilon_x = \frac{y}{\rho} \qquad (5)$$

$\kappa = \frac{1}{\rho}$ is the curvature of the deformed shap.

4.2 Normale Stresses

The normal stress equation is then written according to Hook's law:

$$\sigma_x = E\varepsilon_x = E\frac{y}{\rho}$$

E is the Young's Modulus.

Equilibrium Equations :

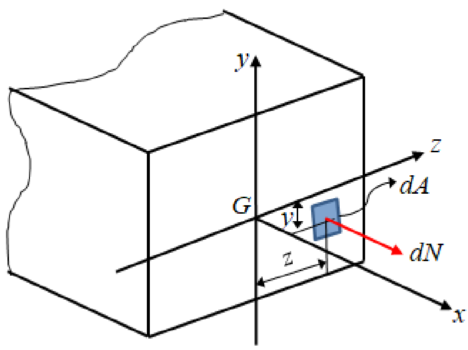


Figure 10

- 1st condition: $N = \int_{(dA)} dN = 0$

$dN = \sigma_x dA = \frac{E}{\rho} y dA$ et donc $N = \int_{(dA)} \frac{E}{\rho} y dA = \frac{E}{\rho} \int_{(dA)} y dA = 0$

The ratio $\boldsymbol{E/\rho}$ does not depend on the dimensions of the section; therefore only the term $\int ydA$ representing the static moment of the section area with respect to the '$\boldsymbol{z}$' axis is zero: $\int ydA = 0$.

Conclusion: the z axis is a central axis, i.e. it passes through the center of gravity of the section.

- Moment about the $\boldsymbol{z}$ – axis

$$dM_z = ydN = y\sigma_x dA = \frac{E}{\rho} y^2 dA \Rightarrow M_z = dM_z = \frac{E}{\rho} \int y^2 dA$$

The term $\int y^2 dA$ represents the quadratic moment of inertia $\boldsymbol{I_z}$.

The normal bending stress equation is written:

$$\sigma_x = E\varepsilon_x = E\frac{y}{\rho} = E\frac{M_z y}{EI_z} = \frac{M_z y}{I_z}$$

Finally, we write: $$\sigma_x = \frac{M_z y}{I_z} \qquad (6)$$

Mz: The bending moment acting on the cross-section;
Iz: The moment of inertia about the z axis;
y: The ordinate of the point under consideration.
Note: $\boldsymbol{\sigma_x}$ is calculated at all points $M(z,y)$ of a cross-section.