Exercise

Given the system of equations:

$$\begin{cases} 4x_1 + x_2 + x_3 + x_4 = 6 \\ x_1 + 5x_2 + x_3 + x_4 = -8 \\ x_1 + x_2 + 6x_3 + x_4 = 10 \\ x_1 + x_2 + x_3 + 7x_4 = -12 \end{cases}$$

1. Verify the convergence condition for iterative methods.
2. Solve the system using the Jacobi method.
3. Solve the system using the Gauss-Seidel method.

Given: Initial guess $\mathbf{X}^{(0)} = (-1, 1, -1, 1)$ and tolerance $\epsilon = 0.01$.

Solution

1. Convergence Condition Verification

The matrix is strictly diagonally dominant since:

$$\begin{cases} |4| > |1| + |1| + |1| = 3 \\ |5| > |1| + |1| + |1| = 3 \\ |6| > |1| + |1| + |1| = 3 \\ |7| > |1| + |1| + |1| = 3 \end{cases}$$

All diagonal elements satisfy $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$, thus both Jacobi and Gauss-Seidel methods will converge.

2. Jacobi Method

Iterative scheme:

$$\begin{cases} x_1^{(k+1)} = (6 - x_2^{(k)} - x_3^{(k)} - x_4^{(k)})/4 \\ x_2^{(k+1)} = (-8 - x_1^{(k)} - x_3^{(k)} - x_4^{(k)})/5 \\ x_3^{(k+1)} = (10 - x_1^{(k)} - x_2^{(k)} - x_4^{(k)})/6 \\ x_4^{(k+1)} = (-12 - x_1^{(k)} - x_2^{(k)} - x_3^{(k)})/7 \end{cases}$$

k	x_1	x_2	x_3	x_4
0	-1.0000	1.0000	-1.0000	1.0000
1	1.2500	-1.8000	1.8333	-1.5714
2	1.8845	-2.0571	2.0306	-1.9429
3	1.9923	-2.0159	2.0041	-1.9974
4	2.0023	-1.9994	1.9996	-2.0003

Converged in 4 iterations with solution:

$$\boxed{x_1 = 2.00}, \boxed{x_2 = -2.00}, \boxed{x_3 = 2.00}, \boxed{x_4 = -2.00}$$

3. Gauss-Seidel Method

Iterative scheme (using new values immediately):

$$\begin{cases} x_1^{(k+1)} = (6 - x_2^{(k)} - x_3^{(k)} - x_4^{(k)})/4 \\ x_2^{(k+1)} = (-8 - x_1^{(k+1)} - x_3^{(k)} - x_4^{(k)})/5 \\ x_3^{(k+1)} = (10 - x_1^{(k+1)} - x_2^{(k+1)} - x_4^{(k)})/6 \\ x_4^{(k+1)} = (-12 - x_1^{(k+1)} - x_2^{(k+1)} - x_3^{(k+1)})/7 \end{cases}$$

k	x_1	x_2	x_3	x_4
0	-1.0000	1.0000	-1.0000	1.0000
1	1.2500	-2.0500	2.0083	-2.0012
2	2.0108	-2.0024	1.9991	-2.0001

Converged in 2 iterations with solution:

$$\boxed{x_1 = 2.00}, \boxed{x_2 = -2.00}, \boxed{x_3 = 2.00}, \boxed{x_4 = -2.00}$$