Chapter VI: Iterative Methods for Linear Systems

While direct methods provide exact solutions to linear systems, they are memory intensive. This chapter introduces iterative (indirect) methods that provide approximate solutions. These methods are easy to implement, memory efficient, and can achieve arbitrary precision.

6.1 Jacobi Method

This method uses the fixed-point principle for linear systems. Given the system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \tag{1}$$

Assuming diagonal elements $a_{ii} \neq 0$, we rewrite as:

$$\begin{cases} x_1 = (b_1 - a_{12}x_2 - \cdots - a_{1n}x_n)/a_{11} \\ x_2 = (b_2 - a_{21}x_1 - \cdots - a_{2n}x_n)/a_{22} \\ \vdots \\ x_n = (b_n - a_{n2}x_2 - \cdots - a_{nn-1}x_{n-1})/a_{nn} \end{cases} \tag{2}$$

The iterative scheme becomes:

$$\begin{cases} x_1^{(k+1)} = (b_1 - a_{12}x_2^{(k)} - \cdots - a_{1n}x_n^{(k)})/a_{11} \\ x_2^{(k+1)} = (b_2 - a_{21}x_1^{(k)} - \cdots - a_{2n}x_n^{(k)})/a_{22} \\ \vdots \\ x_n^{(k+1)} = (b_n - a_{n2}x_2^{(k)} - \cdots - a_{nn-1}x_{n-1}^{(k)})/a_{nn} \end{cases} \tag{3}$$

Starting with initial guess $\mathbf{X}^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$, we compute successive approximations until convergence.

6.2 Gauss-Seidel Method

Improving on Jacobi, this method uses newly computed components immediately:

$$
\begin{cases}
x_1^{(k+1)} = (b_1 - a_{12}x_2^{(k)} - \cdots - a_{1n}x_n^{(k)})/a_{11} \\
x_2^{(k+1)} = (b_2 - a_{21}x_1^{(k+1)} - \cdots - a_{2n}x_n^{(k)})/a_{22} \\
x_3^{(k+1)} = (b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} - \cdots)/a_{33} \\
\vdots \\
x_n^{(k+1)} = (b_n - a_{n2}x_2^{(k+1)} - \cdots - a_{nn-1}x_{n-1}^{(k+1)})/a_{nn}
\end{cases} \tag{4}
$$

6.3 Convergence Conditions

Iterative methods converge if:

1. The spectral radius $\rho(A) < 1$, or equivalently
2. A is strictly diagonally dominant:

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \text{for } i = 1, \ldots, n \tag{5}$$

6.4 Stopping Criterion

The iteration stops when:

$$\|\mathbf{X}^{(k+1)} - \mathbf{X}^{(k)}\|_\infty < \epsilon \tag{6}$$

i.e., when all component-wise differences satisfy $|x_i^{(k+1)} - x_i^{(k)}| < \epsilon$ for $i = 1, \ldots, n$.