Matrix Factorization Methods

5.3 LU Factorization (Crout-Doolittle Method)

Given a system $\boldsymbol{AX} = \boldsymbol{B}$, we factorize $\boldsymbol{A}$ as:

$$\boldsymbol{A} = \boldsymbol{LU}$$

where:

- $\boldsymbol{L}$ is lower triangular ($l_{ij} = 0$ for $j > i$)
- $\boldsymbol{U}$ is upper triangular with $u_{ii} = 1$ ($u_{ij} = 0$ for $i > j$)

Factorization Process

For each i from 1 to n:

1. **Column i of $\boldsymbol{L}$**:

$$l_{ki} = a_{ki} - \sum_{j=1}^{i-1} l_{kj} u_{ji} \quad \text{for } k = i, \ldots, n$$

2. **Row i of $\boldsymbol{U}$**:

$$u_{ik} = \frac{a_{ik} - \sum_{j=1}^{i-1} l_{ij} u_{jk}}{l_{ii}} \quad \text{for } k = i+1, \ldots, n$$

Example Solution

Given system:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}$$

Step 1: LU Factorization

$$\begin{aligned}
i = 1: \quad & l_{11} = a_{11} = 1 \\
& l_{21} = a_{21} = 1 \\
& l_{31} = a_{31} = 2 \\
& u_{12} = \frac{a_{12}}{l_{11}} = 1 \\
& u_{13} = \frac{a_{13}}{l_{11}} = 2 \\
i = 2: \quad & l_{22} = a_{22} - l_{21}u_{12} = 2 - (1)(1) = 1 \\
& l_{32} = a_{32} - l_{31}u_{12} = 1 - (2)(1) = -1 \\
& u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} = \frac{1 - (1)(2)}{1} = -1 \\
i = 3: \quad & l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 1 - (2)(2) - (-1)(-1) = 1 - 4 - 1 = -4
\end{aligned}$$

Thus:

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & -4 \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 2: Forward Substitution ($\boldsymbol{LY} = \boldsymbol{B}$)

$$\begin{cases} 1y_1 = 9 \Rightarrow y_1 = 9 \\ 1y_1 + 1y_2 = 8 \Rightarrow y_2 = -1 \\ 2y_1 - 1y_2 - 4y_3 = 7 \Rightarrow y_3 = \frac{18+1-7}{4} = 3 \end{cases}$$

Step 3: Backward Substitution ($\boldsymbol{UX} = \boldsymbol{Y}$)

$$\begin{cases} 1z = 3 \Rightarrow z = 3 \\ 1y - 1z = -1 \Rightarrow y = 2 \\ 1x + 1y + 2z = 9 \Rightarrow x = 9 - 2 - 6 = 1 \end{cases}$$

Final solution:

$$\boxed{x = 1}, \quad \boxed{y = 2}, \quad \boxed{z = 3}$$

5.4 Cholesky Factorization

For symmetric positive-definite $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{M}\boldsymbol{M}^T$$

where $\boldsymbol{M}$ is lower triangular.

Factorization Process

For $i = 1$ to n:

$$m_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} m_{ik}^2}$$

For $j = i + 1$ to n:

$$m_{ji} = \frac{a_{ij} - \sum_{k=1}^{i-1} m_{ik} m_{jk}}{m_{ii}}$$

Example Solution

Given system:

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$$

Step 1: Cholesky Factorization

$$
\begin{aligned}
i=1: \quad & m_{11} = \sqrt{2} \\
& m_{21} = \frac{1}{\sqrt{2}} \\
& m_{31} = \frac{1}{\sqrt{2}} \\
i=2: \quad & m_{22} = \sqrt{2 - \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \\
& m_{32} = \frac{1 - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}}{\sqrt{\frac{3}{2}}} = \frac{\frac{1}{2}}{\sqrt{\frac{3}{2}}} = \frac{1}{\sqrt{6}} \\
i=3: \quad & m_{33} = \sqrt{2 - \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}
\end{aligned}
$$

Thus:

$$
\boldsymbol{M} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{3}} \end{pmatrix}
$$

Step 2: Forward Substitution ($\boldsymbol{MY} = \boldsymbol{B}$)

$$
\begin{cases}
\sqrt{2}y_1 = 4 \Rightarrow y_1 = 2\sqrt{2} \\
\frac{1}{\sqrt{2}}y_1 + \sqrt{\frac{3}{2}}y_2 = 4 \Rightarrow y_2 = \frac{4\sqrt{6}}{3} \\
\frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 + \frac{2}{\sqrt{3}}y_3 = 4 \Rightarrow y_3 = \sqrt{3}
\end{cases}
$$

Step 3: Backward Substitution ($\boldsymbol{M}^T\boldsymbol{X} = \boldsymbol{Y}$)

$$
\begin{cases}
\frac{2}{\sqrt{3}}z = \sqrt{3} \Rightarrow z = \frac{3}{2} \\
\sqrt{\frac{3}{2}}y + \frac{1}{\sqrt{6}}z = \frac{4\sqrt{6}}{3} \Rightarrow y = \frac{3}{2} \\
\sqrt{2}x + \frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z = 2\sqrt{2} \Rightarrow x = 1
\end{cases}
$$

Final solution:

$$
\boxed{x = 1}, \quad \boxed{y = \frac{3}{2}}, \quad \boxed{z = \frac{3}{2}}
$$