5.3 Crout-Doolittle (LU) Method

This method factorizes matrix A into:

- L: Lower triangular matrix
- U: Upper triangular matrix with diagonal elements $u_{ii} = 1$

For system $AX = B$, we compute $A = LU$ then solve:

$$\begin{cases} LY = B & \text{(Forward substitution)} \\ UX = Y & \text{(Backward substitution)} \end{cases}$$

5.3.1 Matrix Determination

$$L = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & u_{12} & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Elements are computed by:

$$\begin{cases} l_{ki} = a_{ki} - \sum_{j=1}^{i-1} l_{kj} u_{ji} \\ u_{ik} = \frac{a_{ik} - \sum_{j=1}^{i-1} l_{ij} u_{jk}}{l_{ii}} \end{cases} \quad \text{for } i = 2, \ldots, n \text{ and } k = i, \ldots, n$$

Example Solution

Given:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}$$

Step 1: LU Factorization

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 2: Solve $LY = B$

$$\begin{cases} y_1 = 9 \\ y_1 + y_2 = 8 \Rightarrow y_2 = -1 \\ 2y_1 - y_2 + y_3 = 7 \Rightarrow y_3 = -10 \end{cases}$$

Step 3: Solve $UX = Y$

$$\begin{cases} x + y + 2z = 9 \\ y - z = -1 \Rightarrow y = -1 + z \\ z = -10 \end{cases}$$

Final solution:

$$x = 1, \quad y = -11, \quad z = -10$$

5.4 Cholesky Method

For symmetric positive-definite matrices ($A = A^T$, $\det(A) > 0$), factorize as:

$$A = MM^T$$

where M is lower triangular.

Matrix Determination

$$M = \begin{pmatrix} m_{11} & 0 & \cdots & 0 \\ m_{21} & m_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}$$

Elements computed by:

$$\begin{cases} m_{ii} = \sqrt{a_{ii} - \sum_{j=1}^{i-1} m_{ij}^2} \\ m_{ji} = \frac{a_{ij} - \sum_{k=1}^{i-1} m_{ik} m_{jk}}{m_{ii}} \end{cases}$$

Example Solution

Given:

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$$

Step 1: Cholesky Factorization

$$M = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{3}} \end{pmatrix}$$

Step 2: Solve $MY = B$

$$\begin{cases} \sqrt{2} y_1 = 4 \Rightarrow y_1 = 2\sqrt{2} \\ \frac{1}{\sqrt{2}} y_1 + \sqrt{\frac{3}{2}} y_2 = 4 \Rightarrow y_2 = \frac{4\sqrt{6}}{3} \\ \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{6}} + \frac{2y_3}{\sqrt{3}} = 4 \Rightarrow y_3 = \sqrt{3} \end{cases}$$

Step 3: Solve $M^T X = Y$

$$\begin{cases} \frac{2}{\sqrt{3}} z = \sqrt{3} \Rightarrow z = 1.5 \\ \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{6}} = \frac{4\sqrt{6}}{3} \Rightarrow y = 1.5 \\ \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{2}} = 2\sqrt{2} \Rightarrow x = 1 \end{cases}$$

Final solution:

$$x = 1, \quad y = 1.5, \quad z = 1.5$$