1. *Explain why f is of class C^1 on the open set $\mathbb{R}^* \times \mathbb{R}^*$.*

2. *Study the derivability at 0 of the function*

$$g : \mathbb{R} \to \mathbb{R},\ x \mapsto g(x) = f(x, x).$$

   Deduce that the function f is not of class C^1 on $\mathbb{R}^2$.

Exercice 12 *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by :*

$$f(x, y, z) = x^3 y + x^2 - y^2 - x^4 + z^5.$$

Calculate the Hessian matrix of f.

Exercice 13 *Let $\theta : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function that satisfies $\dfrac{\partial \theta}{\partial x}(1,2) = -1$ and $\dfrac{\partial \theta}{\partial y}(1,2) = 3$. We define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(t) := \theta(t^2, 3t-1)$. Calculate $f'(1)$.*

Exercice 14 *Using the chain rule, calculate the derivative of the function $w : \mathbb{R} \to \mathbb{R}$, where*

$$\begin{aligned} w(t) &= f\left(x(t), y(t), z(t)\right), \\ f\left(x, y, z\right) &= x\,\mathrm{e}^{\frac{y}{z}}, \\ x(t) &= t^2, \qquad y(t) = 1, \qquad z(t) = 1 + 2t. \end{aligned}$$

Exercice 15 *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a function of class C^1 and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be the function*

$$g(u, v) = f(\cos u + \sin v, \sin u + \cos v, \mathrm{e}^{u-v}).$$

1. *Show that g is of class C^1.*

2. *We assume that the Jacobian matrix of f at the point $a = (1, 1, 1)$ is*

$$\mathbf{J}(f)_a = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \end{pmatrix}.$$

   Determine the differential of g at point $b = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Exercice 16 (Spherical coordinates) *Let the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by*

$$f(\rho, \theta, \varphi) = \left(\rho \cos\theta \sin\varphi, \rho \sin\theta \sin\varphi, \rho \cos\varphi\right).$$

1. *Calculate the partial derivatives of f with respect to the variables ρ, θ and φ.*

2. *Determine the Jacobian matrix as well as the Jacobian of f.*

3. *Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function on $\mathbb{R}^3$. Calculate the partial derivatives of the composite function $\psi = g \circ f$ as a function of the partial derivatives of the function g.*

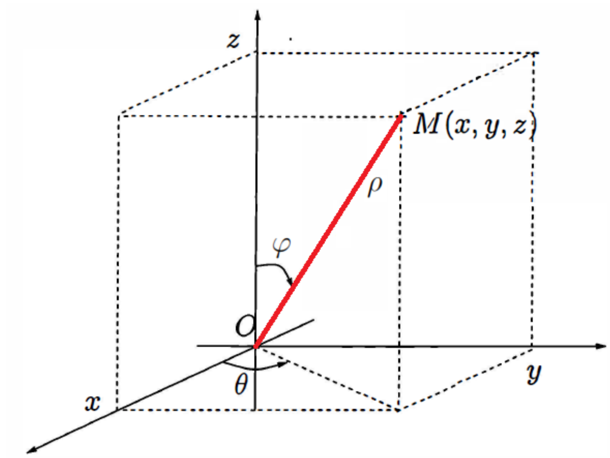


Illustration of the transition to spherical coordinates

Exercice 17 *Let f be a function of class C^1 on $\mathbb{R}^2$. Calculate the derivatives (possibly partial) of the following functions:*

1. $g(x, y) = f(y, x)$,
2. $g(x) = f(x, x)$,
3. $g(x, y) = f(y, f(x, x))$,
4. $g(x) = f(x, f(x, x))$.

Exercice 18 *Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x^3 + 3x\,e^y, y - x^2)$ is a C^1-diffeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$.*

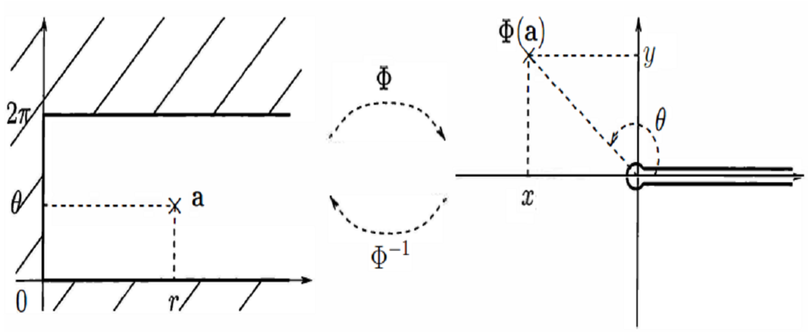
Exercice 19 *We denote $U =]0, +\infty[^2$ and $\varphi : (x, y) \mapsto \left(x^3y^2, \dfrac{1}{x^2y}\right)$.*
Show that φ is a $C^\infty-$diffeomorphism of U onto U.

Exercice 20 *Let the function $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta)\,.$$

1. *Show that Φ is not a C^1-diffeomorphism on $\mathbb{R}^2$.*
2. *Show that if we restrict the function Φ to the open $U =]0, +\infty[\, \times\,]0, 2\pi[$, then we obtain an injective application.*

3. Deduce that Φ is a C^1-diffeomorphism of U onto $V = \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 \ ; \ x \geq 0\}$.



Exercice 21 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by:*

$$f(x,y) = \sin x \sin y.$$

Write the Taylor-Young expansion of order 2 of f in the neighborhood of the point $(0,0)$.

Exercice 22 *Let $f :]-1,1[^2 \to \mathbb{R}$ be the function defined by*

$$f(x,y) = -\frac{1+x}{((1+x)^2+y^2)^{3/2}}.$$

Write the Taylor-Young expansion of order 2 of f in the neighborhood of $(0,0)$.

Exercice 23 *Show that there exists a neighborhood V of $(0,0)$ in $\mathbb{R}^2$ and a unique function $\varphi : V \to \mathbb{R}$ such that: φ is of class C^∞ on V, $\varphi(0,0) = 1$, and, for all $(x,y) \in V$, $\varphi(x,y)$ is a solution to the equation $z^5 + xz^2 + yz - 1 = 0$, with unknown $z \in \mathbb{R}$.*

Exercice 24 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ of class C^1 such that:*

$$f(0,0) = 0, \quad \frac{\partial f}{\partial x}(0,0) \neq -1, \quad \frac{\partial f}{\partial y}(0,0) \neq 0.$$

Show that the relation $f(f(x,y),y) = 0$ implicitly defines y as a function of x in the neighborhood of $(0,0)$.