University of Oum El Bouaghi — Academic year: 2024/2025

Analysis 4 — License 2 - Mathematics

Sheet of exercises N°3 - Differential calculus

Exercice 1

1. *Show that the function* $f : (x, y) \in \mathbb{R}^2 \mapsto \sin(x+y)$ *admits directional derivative according to the two vectors* $\mathbf{u} = (-1, 1)$ *and* $\mathbf{v} = (1, 1)$ *at every point* $a \in \mathbb{R}^2$ *and calculate* $\dfrac{\partial f}{\partial \mathbf{u}}(a)$ *and* $\dfrac{\partial f}{\partial \mathbf{v}}(a)$.
2. *Let* g *be a function from* $\mathbb{R}$ *to* $\mathbb{R}$ *differentiable on* $\mathbb{R}$ *and* f *the function* $f : (x, y) \in \mathbb{R}^2 \mapsto g(x + y)$. *Determine the derivatives of* f *in all directions* $\mathbf{v} \in \mathbb{R}^2 \setminus \{(0, 0)\}$ *and at every point* $a \in \mathbb{R}^2$.

Exercice 2 *Write the Jacobian matrix of the function* $f : (x, y, z) \in \mathbb{R}^3 \mapsto (xyz, x^2y + y)$ *at every point of* $\mathbb{R}^3$.

Exercice 3 *Calculate the differential of the function* $f : (x, y) \mapsto 3x^2y - 4xy$ *at the point* $a = (1, 2)$.
Deduce the directional derivative of f *at the point* $a = (1, 2)$ *along the direction* $\mathbf{v} = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

Exercice 4 *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

1. *Calculate the first partial derivatives of* f.
2. *Determine the Jacobian matrix of* f *at the point* $(1, 1)$ *then deduce the expression of the differential of* f *at* $(1, 1)$.

Exercice 5 *Form the equations of the tangent plane and the normal to the surface* $z = \dfrac{x^2}{2} - y^2$ *at the point* $(2, -1, 1)$.

Exercice 6 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by*

$$f(x, y) = x\,\mathrm{e}^{xy}.$$

Is f *differentiable at the point* $(1, 0)$ *? If yes, linearize* f *in the neighborhood of* $(1, 0)$ *and approach the value* $f(1.1, -0.1)$.

Exercice 7 *The height of a cone is* $H = 30$ cm, *the radius of its base is* $R = 10$ cm. *Study the variation of the volume of the cone if we increase* H *by* 3 mm *and decrease* R *by* 1 mm. *Indication: the volume of the cone is given by the relation* $V = \frac{1}{3}\pi R^2 H$.

Exercice 8 *One side of a rectangle is* $a = 10$ cm, *the other* $b = 24$ cm. *Study the variation of the diagonal* l *of the rectangle if side* a *lengthens by* 4 mm *and side* b *shortens by* 1 mm. *Calculate the approximate value of the variation and compare with the exact value.*

Exercice 9 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by*

$$f(x,y) = \begin{cases} \frac{x^2y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{else.} \end{cases}$$

1. *Explain why* f *is of class* C^1 *on* $\mathbb{R}^2 \setminus \{(0,0)\}$.
2. *Is* f *continuous on* $\mathbb{R}^2$ *? (We can switch to polar coordinates)*
3. *Does* f *admit partial derivatives of order* 1 *on* $\mathbb{R}^2$ *?*
4. *Is* f *of class* C^1 *on* $\mathbb{R}^2$ *? (We can switch to polar coordinates)*
5. *Is* f *differentiable on* $\mathbb{R}^2$ *?*

Exercice 10 *Consider the function* f *from* $\mathbb{R}^2$ *to* $\mathbb{R}$ *defined by*

$$f(x,y) = \begin{cases} y^2 \sin\left(\dfrac{x}{y}\right) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

1. *Show that* f *is of class* C^1 *on the open set* $U = \{(x,y) \in \mathbb{R}^2 \; ; \, y \neq 0\}$.
2. *Show that* f *is continuous on* $\mathbb{R}^2$.
3. *Show that* f *is differentiable on* $\mathbb{R}^2$.
4. *Calculate the two partial derivatives* $\dfrac{\partial f}{\partial x}$ *and* $\dfrac{\partial f}{\partial y}$ *on* $\mathbb{R}^2$. *Deduce that* f *is not of class* C^1 *on* $\mathbb{R}^2$.
5. *Calculate* $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$ *and* $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$.

Exercice 11 *Consider the function* f *defined on* $\mathbb{R}^2$ *by:*

$$f(x,y) = \frac{\sin(xy)}{|x|+|y|} \quad \text{if } (x,y) \neq (0,0) \qquad \text{and} \qquad f(0,0) = 0.$$