Chapter 2

Linear Applications

2.1 Concept of Linear Application

Definition 2.1.1

1. Assume that $(E,+,.)$ and $(F,+,.)$ are two K-vector spaces. Define f as a function that maps from E to F. f is considered a linear application if and only if :

$$\forall x, y \in E, \forall \lambda \in \mathbb{R}, \quad f(x+y) = f(x) + f(y) \quad \text{and} \quad f(\lambda x) = \lambda f(x)$$

   Or, in the equivalent case :

$$\forall x, y \in E, \forall \lambda, \mu \in \mathbb{R}, \quad f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$$

2. Furthermore, we refer to f as an isomorphism from E to F if f is bijective.
3. A linear application from $(E,+,.)$ to $(E,+,.)$ is called an endomorphism.
4. An isomorphism from $(E,+,.)$ to $(E,+,.)$ is also called an automorphism of E.

Example 2.1.2

1. The function

$$\begin{aligned} f_1 : \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto x - y, \end{aligned}$$

   is a linear application, as :

$$\begin{aligned} f_1(\lambda(x,y) + \mu(x',y')) &= f_1((\lambda x, \lambda y) + (\mu x', \mu y')) \\ &= f_1((\lambda x + \mu x'), (\lambda y + \mu y')) \\ &= (\lambda x + \mu x') - (\lambda y + \mu y') \\ &= \lambda x - \mu y + \mu x' - \mu y', \\ &= \lambda(x - y) + \mu(x' - y'), \\ &= \lambda f_1(x,y) + \mu f_1(x',y'). \end{aligned}$$

2. The function

$$\begin{aligned} f_2 : \mathbb{R}^3 &\to \mathbb{R}^3 \\ (x,y,z) &\mapsto (-x + y, x - 5z, y), \end{aligned}$$

is a linear application, as :

$$\begin{aligned} f_2(\lambda(x,y,z)+\mu(x',y',z')) &= f_2((\lambda x,\lambda y,\lambda z)+(\mu x',\mu y',\mu z')) \\ &= f_2((\lambda x+\mu x'),(\lambda y+\mu y'),(\lambda z+\mu z')) \\ &= (-(\lambda x+\mu x')+(\lambda y+\mu y'),(\lambda x+\mu x')-5(\lambda z+\mu z'),\lambda y+\mu y') \\ &= (-\lambda x-\mu x'+\lambda y+\mu y',\lambda x+\mu x'-5\lambda z-5\mu z',\lambda y+\mu y') \\ &= (-\lambda x+\lambda y-\mu x'+\mu y',\lambda x-5\lambda z+\mu x'-5\mu z',\lambda y+\mu y') \\ &= (\lambda(-x+y)+\mu(-x'+y'),\lambda(x-5z)+\mu(x'-5z'),\lambda y+\mu y') \\ &= \lambda(-x+y,x-5z,y)+\mu(-x'+y',x'-5z',y'), \\ &= \lambda f_2(x,y,z)+\mu f_2(x',y',z'). \end{aligned}$$

3. The function

$$\begin{aligned} f_3 : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto -5x, \end{aligned}$$

is an isomorphism. Indeed, f_3 is linear because for all x, y in $\mathbb{R}$, and λ, μ in $\mathbb{R}$:

$$\begin{aligned} f_3(\lambda x+\mu y) &= -5(\lambda x+\mu y), \\ &= \lambda f_3(x)+\mu f_3(y). \end{aligned}$$

Note 2.1.3

It can be easily shown that the sum of two linear applications is linear. Also, the product of a linear application by a scalar and the composition of two linear applications are linear.

Proposition 2.1.4

Let f be a linear application from E to F.

1. $f(O_E)=O_F$.
2. $\forall x\in E$, $f(-x)=-f(x)$.

Proof

Because f is a linear application we have :

1. $f(O_E)=f(O_E+O_E)=f(O_E)+f(O_E)\Rightarrow f(O_E)-f(O_E)=f(O_E)$. So, $O_F=f(O_E)$.
2. Since f is a linear application, then $\forall x\in E$, $f(-x)+f(x)=f(-x+x)=f(O_E)=O_F$ then, $f(-x)=-f(x)$.

2.1.1 Image and Kernel of a Linear Application

Definition 2.1.5

1. The definition of the image of f, or $Im\ f$, is as follows :

$$Im\ f=\{y\in F,\quad \exists x\in E: f(x)=y\}=\{f(x),\quad x\in E\}.$$

2. The following is the definition of the kernel of f, or $ker\ f$:

$$ker\ f=\{x\in E,\quad f(x)=O_F\}.$$

Sometimes $ker\ f$ is denoted by : $f^{-1}(\{0\})$.

2.1.2 Rank of a Linear Application

Proposition 2.1.6

If f is a linear application from E to F, and if $dim\ Im\ f = n < +\infty$, then n is called the rank of f and is denoted $rg(f)$. $Im\ f$ and $ker\ f$ are vector subspaces of E.

Definition 2.1.7

The dimension of the image of f is the definition of the rank of f :

$$rg(f) = dim\ Im\ f.$$

Example 2.1.8

1. Determine the kernel of the function f_1 :

$$f_1(x, y) = x + 2y$$

The kernel is given by :

$$\begin{aligned} ker\ f_1 &= \{(x, y) \in \mathbb{R}^2 / f_1(x, y) = 0\} \\ &= \{(x, y) \in \mathbb{R}^2 / x + 2y = 0\} \\ &= \{(x, y) \in \mathbb{R}^2 / x = -2y\} \end{aligned}$$

So, $ker\ f_1$ is a line in $\mathbb{R}^2$.

2. Determine the kernel of the function f_2 :

$$f_2(x, y, z) = (-x + y, x - 5z, y)$$

The kernel is given by :

$$\begin{aligned} ker\ f_2 &= \{(x, y, z) \in \mathbb{R}^3 / f_2(x, y, z) = (0, 0, 0)\} \\ &= \{(x, y, z) \in \mathbb{R}^3 / -x + y = 0, x - 5z = 0, y = 0\} \end{aligned}$$

Solving this system of equations gives the solution space, which is a subspace of $\mathbb{R}^3$.

Proposition 2.1.9

Let f be a linear application from E to F. The following are equivalent :

1. f is surjective $\Leftrightarrow Im\ f = F$.
2. f is injective $\Leftrightarrow ker\ f = \{0_E\}$.

Example 2.1.10

In the previous example $Im\ f_2 = \mathbb{R}^3$ then f_2 is surjective. We are going to demonstrate that f_2 is injective

$ker\ f_2 = \{(x, y, z) \in \mathbb{R}^3,\ \ f_2(x, y, z) = (0, 0, 0)\}$,

$ker\ f_2 = \{(x, y, z) \in \mathbb{R}^3, (-x + y, x - z, y) = (0, 0, 0)\}$

$$\begin{cases} -x + y = 0 \\ x - z = 0 \\ y = 0 \end{cases} \Rightarrow \begin{cases} -x + 0 = 0 \\ x - z = 0 \\ y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ z = 0 \\ y = 0 \end{cases}$$

Therefore, $ker\ f_2 = \{(0,0,0)\}$, hence f_2 is injective. So f_2 is bijective.

2.2 Finite dimension and rank theorem

Proposition 2.2.1

Let f, g be two linear mappings from E to F, such as E and F be two K vector spaces. If E is of finite dimension n and $\{e_1, e_2, ..., e_n\}$ is a basis of E. Then
$\forall k \in \{1, 2, .., n\},\ f(e_k) = g(e_k) \Leftrightarrow \forall x \in E,\ f(x) = g(x)$.

Proof

The implication ($\Leftarrow$) is obvious. For ($\Rightarrow$)
Since E is generated by $\{e_1, e_2, ..., e_n\}$, for any $x \in E$, there exist λ_1,λ_2, ..., $\lambda_n \in \mathbb{K}$ such that $x = \lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n$,
As f and g are linear, we have
$f(x) = f(\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n) = \lambda_1 f(e_1) + \lambda_2 f(e_2) + ... + \lambda_n f(e_n)$,
$g(x) = g(\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n) = \lambda_1 g(e_1) + \lambda_2 g(e_2) + ... + \lambda_n g(e_n)$,
so if we assume that $\forall k \in \{1, 2, .., n\},\ f(e_k) = g(e_k)$, then we deduce that
$\forall x \in E,\ f(x) = g(x)$.

Note 2.2.2

For two linear mappings f and g from E to F to be equal, it is sufficient that they coincide on the basis of the $\mathbb{K}$ vector space E.

Example 2.2.3

Let g be a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that
$g(1,0) = (2,1),\ \ g(0,1) = (-1,-1)$.
Then let's determine the value of g at all points of $\mathbb{R}^2$, in fact, we have :

$\forall (x,y) \in \mathbb{R}^2, (x,y) = x(1,0) + y(0,1)$.

$\forall (x,y) \in \mathbb{R}^2$,
$g(x,y) = g(x(1,0) + y(0,1)) = xg(1,0) + yg(0,1)$
$= x(2,1) + y(-1,-1)$
$= (2x, x) + (-y, -y)$
$= (2x - y, x - y)$.
So, $g(x,y) = (2x - y, x - y)$.

Theorem 2.2.4

(**Rank Theorem**)
Let E be a vector space of finite dimension and f a linear mapping from E to F. Next, there is the equality.

$$\boxed{dim\ (Im\ f\ ) + dim\ (\ Ker\ f\ \ ) = dim(E\ )}$$

Another way to express the rank theorem is as

$$\boxed{rg(f\ ) + dim\ (Ker\ f\ ) = dim(E\ )}$$

Example 2.2.5

We have shown that $dim\ ker\ f_1 = 1$ for f_1 defined by
$f_1 : \mathbb{R}^2 \to \mathbb{R}$
$(x, y) \mapsto x + 2y$
Since $dim\ \mathbb{R}^2 = 2$ then, $dim\ Im\ (f_1) = dim\ \mathbb{R}^2\ - dim\ ker\ f_1 = 2 - 1 = 1$.

Note 2.2.6

1. Always , the rank of f is either the same as or less than the dimension of E.
2. Additionally, if F has a finite dimension, we have $rg(f) \leq dim(F)$. Indeed, $Im(f)$ is a subspace of F, and therefore its dimension (equal to $rg(f)$ by definition) is less than or equal to that of F.
3. In the case when F has a finite dimension, then f is surjective if and only if $rg\ (f) = dim\ (F)$. Indeed, $Im(f)$ is a subspace of F, and therefore $Im(f) = F$ if and only if $dim(Im(f)) = dim(F)$.

Corollary 2.2.7

Assuming that E is of finite dimension. Let f be a linear mapping from E to F.

1. f is an injective linear application if and only if $rg(f) = dim(E)$. Then we have :
$$dim(E) \leq dim(F).$$
2. When f is surjective, then F is of finite dimension and
$$dim(F) \leq dim(E).$$
3. When f is bijective, then F is of finite dimension and
$$dim(F) = dim(E)$$

2.2.1 Rank theorem if $dim(E) = dim(F)$

A very important consequence of the rank theorem is the following :

Theorem 2.2.8

The following equivalencies hold if F has a finite dimension and $dim(E) = dim(F) = n$.

$$f \text{ injective} \Leftrightarrow f \quad \text{surjective} \Leftrightarrow f \quad \text{bijective} \Leftrightarrow f \quad \text{isomorphism}$$

This result applies in particular to endomorphisms.

Example 2.2.9

1. The mapping f_1 is not an isomorphism because $dim\ \mathbb{R}^2 \neq dim\ \mathbb{R}$.
2. Let $g(x, y) = (2x - y, x - y)$, g defined from $\mathbb{R}^2$ to $\mathbb{R}^2$.
We have g is an isomorphism ($\dim\mathbb{R}^2 = \dim \mathbb{R}^2 = 2$)
because $dim\ ker\ g = 0$ indeed :
$$ker\ g = \{(x, y) \in \mathbb{R}^2, (2x - y, x - y) = (0, 0)\} = \{(0, 0)\},$$
it is even an automorphism.

2.2.2 Composition of linear applications

Proposition 2.2.10

The composition of two linear mappings is still linear. More formally, it can be stated as :
$\forall p, q, r \in \mathbb{N}, \forall f \in Lq, r, \forall g \in Lp, q \ , \ g \circ f$ is linear.

Note 2.2.11

Recall that Lp, q designates the set of linear mappings from $\mathbb{R}^q$ to $\mathbb{R}^p$.

$$\begin{array}{ll} \circ_{p,q,r} & : L_{p,q} \times L_{q,r} \to L_{p,r} \\ (g, f) & \mapsto g \circ f, \\ (g, f) & \mapsto (v \mapsto g(f(v)). \end{array}$$

It should be seen as follows :

$$\mathbb{R}^r \xrightarrow{f} \mathbb{R}^q \xrightarrow{g} \mathbb{R}^p$$

Proof

Let p, q, r be three integers, f in Lq, r and g in Lp, q. In order to demonstrate that $g \circ f$ is linear, we must :
$\forall \lambda, \mu \in \mathbb{R}, \forall u, v \in \mathbb{R}^r$, $(g \circ f)(\lambda u + \mu v) = \lambda(g \circ f)(u) + \mu(g \circ f)(v)$
We have : $(g \circ f)(\lambda u + \mu v) = g(f(\lambda u + \mu v))$ (by definition of composition)
$= g(\lambda f(u) + \mu f(v))$ (by linearity of f)
$= \lambda g(f(u)) + \mu g(f(v))$ (by linearity of g)

Example 2.2.12

Determine the composition $g \circ f$ using

$$g : \begin{array}{ll} \mathbb{R}^2 & \to \mathbb{R}^2 \\ (x, y) & \mapsto (x + y, \ x - y) \end{array}$$

$$f : \begin{array}{ll} \mathbb{R}^3 & \to \mathbb{R}^2 \\ (x, y, z) & \mapsto (3x + 5y + 7z, \quad 2x + 2y + 2z) \end{array}$$

$$\begin{array}{rcl} (g \circ f)(x, y, z) & = & (g(f(x, y, z)), \text{ by using composition definition} \\ & = & g(3x + 5y + 7z, 2x + 2y + 2z), \text{ by definition of } f \\ & = & (5x + 7y + 9z, x + 3y + 5z), \text{ by definition of } g \end{array}$$

The composition $g \circ f$ is $(x, y, z) \mapsto (5x + 7y + 9z, \ \ x + 3y + 5z)$.

Proposition 2.2.13

The composition of linear applications is associative. More formally, this can be expressed as :
$\forall p, q, r, s \in \mathbb{N}, \forall h \in Lp, q, \forall g \in Lq, r, \forall f \in Lr, s, (h \circ g) \circ f = h \circ (g \circ f)$.

Proof

Let $p, q, r, s, \ f \ , \ g \ , \ h$ ´We are going to prove
$(h \circ g) \circ f = h \circ (g \circ f)$ we say that
$\forall u \in \mathbb{R}^s$, $((h \circ g) \circ f)(u) = (h \circ (g \circ f))(u)$.

Let then u in $\mathbb{R}^s$. By usnig the definition of the composition we get

$$((h \circ g) \circ f)(u) = (h \circ g)(f(u)) = h(g(f(u))).$$

The same $(h \circ (g \circ f))(u) = h((g \circ f)(u)) = h(g(f(u)))$.
Then, $((h \circ g) \circ f)(u) = (h \circ (g \circ f))(u)$.

2.2.3 Inverse of a Bijective Linear application, Automorphism.

Theorem 2.2.14

When f be an isomorphism from E to F, then, f^{-1} is an isomorphism from F to E.

Proposition 2.2.15

When f be an automorphism from E to F, then, f^{-1} is an automorphism from F to E.
Assume that E has two automorphisms : f and$g$. Then, $g \circ f$ is an automorphism of E, and we have :

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$