1.3 Solution of exercises

Solution 1.2.1

1. F is a subspace vector space (s.e.v.) if and only if :

$$\begin{cases} F \neq \phi \\ \forall X, Y \in F, \forall \lambda, \mu \in \mathbb{R},\ \lambda X + \mu Y \in F. \end{cases}$$

   (a) $0_{\mathbb{R}^3} = (0,0,0) \in F \Leftrightarrow F \neq \phi$ because $2 \cdot 0 + 0 - 0 = 0$.

   (b) For all $X = (x,y,z)$, $Y = (x',y',z') \in F$, and $\lambda, \mu \in \mathbb{R}$, we want to show that :

$$\lambda(x,y,z) + \mu(x',y',z') \overset{?}{\in} F,$$

   i.e.,

$$(\lambda x + \mu x', \lambda y + \mu y', \lambda z + \mu z') \overset{?}{\in} F.$$

   $2(\lambda x + \mu x') + (\lambda y + \mu y') - (\lambda z + \mu z') = \lambda(2x + y - z) + \mu(2x' + y' - z') = \lambda \cdot 0 + \mu \cdot 0 = 0,$
   since $(x,y,z) \in F \Rightarrow 2x + y - z = 0$ and $(x',y',z') \in F \Rightarrow 2x' + y' - z' = 0$.
   Thus, $\lambda(x,y,z) + \mu(x',y',z') \in F$, and F is a subspace vector space of $\mathbb{R}^3$.

2. Basis of F : Let $X \in F$, i.e., $2x + y - z = 0 \Rightarrow z = 2x + y$. $X = (x,y,z) = (x,y,2x+y) = x(1,0,2) + y(0,1,1)$, so

$$F = \{(x,y,z) \in \mathbb{R}^3 \mid 2x + y - z = 0\} = \{x(1,0,2) + y(0,1,1) \mid x, y \in \mathbb{R}\}.$$

   Hence, F is generated by $\{v_1 = (1,0,2), v_2 = (0,1,1)\}$. Let's show that this family is linearly independent if and only if :

$$\forall \lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 v_1 + \lambda_2 v_2 = (0,0,0) \Rightarrow \lambda_1 = \lambda_2 = 0.$$

   $\lambda_1(1,0,2) + \lambda_2(0,1,1) = (0,0,0) \Rightarrow (\lambda_1, \lambda_2, 2\lambda_1 + \lambda_2) = (0,0,0)$.
   This implies $\lambda_2 = -\lambda_1$ and $\lambda_1 = \lambda_2 = 0$.
   Therefore, $\{v_1, v_2\}$ is a basis (linearly independent and generating) of $\mathbb{R}^3$.

3. $F \neq \mathbb{R}^3$ since $\dim F = 2 \neq 3 = \dim \mathbb{R}^3$.

Solution 1.2.2

1. $(0,0,0) \in F$ because $(0,0,0) = (0-0, 2\times 0 + 0 + 4 \times 0, 3 \times 0 + 2 \times 0)) \Rightarrow F \neq \phi$.
   $\forall X, Y \in F$, λ, $\mu \in \mathbb{R}$, let's show that $\lambda X + \mu Y \overset{?}{\in} F$
   $X \in F$, $\exists (x,y,z) \in \mathbb{R}^3 / \quad X = (x - y, 2x + y + 4z, 3y + 2z)$
   $Y \in F$, $\exists (x',y',z') \in \mathbb{R}^3 / \quad Y = (x' - y', 2x' + y' + 4z', 3y' + 2z')$
   $\lambda\ X + \mu Y = (\lambda x - \lambda y + \mu x' - \mu y', 2\lambda x + \lambda y + 4\lambda z + 2\mu x' + \mu y' + 4\mu z', 3\lambda y + 2\lambda z + 3\mu y' + 2\mu z') +$
   $(\lambda x - \lambda y + \mu x' - \mu y', 2\lambda x + \lambda y + 4\lambda z + 2\mu x' + \mu y' + 4\mu z', 3\lambda y + 2\lambda z + 3\mu y' + 2\mu z')$
   $= ((\lambda x + \mu x') - (\lambda y + \mu y'), 2(\lambda x + \mu x') + (\lambda y + \mu y') + 4(\lambda z + \mu z'), 3(\lambda y + 2\mu z) + 2\lambda z + 2\mu z')$
   $= ((\lambda x + \mu x') - (\lambda y + \mu y'), 2(\lambda x + \mu x') + (\lambda y + \mu y') + 4(\lambda z + \mu z'), 3(\lambda y + 2\mu z) + 2(\lambda + \mu) z')$
   $\exists x'' = (\lambda x + \mu x')$, $\exists y'' = (\lambda y + \mu y')$, $\exists z'' = (\lambda z + \mu z')$, thus
   $\lambda X + \mu Y = (x'' - y'',\ 2x'' + y'' + 4z'',\ 3y'' + 2z'') \in F$.

2. Basis of F :
   Let $X \in F$, $\exists (x,y,z) \in \mathbb{R}^3 / \quad X = (x - y,\ 2x + y + 4z,\ 3y + 2z)$,
   $X = (x - y, 2x + y + 4z, 3y + 2z) = x(1,2,0) + y(-1,1,3) + y(0,4,2)$, thus
   $F = \{x(1,2,0) + y(-1,1,3) + y(0,4,2)/x, y \in \mathbb{R}\}$.
   Therefore, F is generated by $\{v_1 = (1,2,0), v_2 = (-1,1,3), v_3 = (0,4,2)\}$,
   let's show that this family is linearly independent if and only if
   $\forall\ \lambda_1, \lambda\ _2,\ \lambda_3 \in \mathbb{R}$, $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = (0,0,0) \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$.
   $\lambda_1(1,2,0) + \lambda_2(-1,1,3) + \lambda_3(0,4,2) = (0,0,0) \Rightarrow (\ \lambda_1,\ \lambda_2, -\ \lambda_1, -\lambda_2) = (0,0,0)$
   Thus the dimension of F is 3, as $\{v_1, v_2, v_3\}$ is a basis (linearly independent and generating) of $\mathbb{R}^3$.

3. $F = \mathbb{R}^3$ because $\dim F = 3 = \dim \mathbb{R}^3$.

Solution 1.2.3

1. • $(0,0,0,0) \in F \Rightarrow F \neq \phi$ because $(0+0=0) \wedge (0+0=0)$.
   • $\forall X = (x,y,z,t)$, $Y = (x',y',z',t') \in F$, λ, $\mu \in \mathbb{R}$, let's show that :
   $\lambda(x,y,z,t) + \mu(x',y',z',t') \overset{?}{\in} F$
   We have :
   $$\begin{cases} X \in F \Rightarrow (x+z=0) \wedge (y+t=0) \\ Y \in F \Rightarrow (x'+z'=0) \wedge (y'+t'=0) \end{cases}$$
   $$\Rightarrow \begin{cases} \lambda(x+z) = 0 \wedge \mu(x'+z') = 0 \\ \wedge \\ \lambda(y+t) = 0 \wedge \mu(y'+t') = 0 \end{cases}$$
   $$\Rightarrow \begin{cases} \lambda x + \mu x' + \lambda z + \mu z' = 0 \\ \wedge \\ \lambda y + \mu t + \lambda t + \mu t' = 0 \end{cases}$$
   Therefore :
   $(\lambda x + \mu x' + \lambda z + \mu z') = 0 \wedge (\lambda y + \mu t + \lambda t + \mu t') = 0$
   which means
   $\lambda(x,y,z,t) + \mu(x',y',z',t') \in F$ hence the result.
2. Basis of F :
   For $X \in F$, $x = -z \wedge y = -t$,
   $X = (x,y,z,t) = (x,y,-x,-y) = x(1,0,-1,0) + y(0,1,0,-1)$
   $F = \{x(1,0,-1,0) + y(0,1,0,-1)/x,y \in \mathbb{R}\}$.
   Therefore, F is generated by $\{v_1 = (1,0,-1,0),\ \ v_2 = (0,1,0,-1)\}$, let's show that this family is linearly independent if and only if
   $\forall\ \lambda_1,\ \lambda_2 \in \mathbb{R}$, $\lambda_1 v_1 + \lambda_2 v_2 = (0,0,0,0) \Rightarrow \lambda_1 = \lambda_2 = 0$.
   $\lambda_1(1,0,-1,0) + \lambda_2(0,1,0,-1) = (0,0,0,0) \Rightarrow (\ \lambda_1,\ \lambda_2, -\ \lambda_1, -\lambda_2) = (0,0,0,0)$
   Therefore, the dimension of F is 2, as $\{v_1, v_2\}$ is a basis (linearly independent and generating) of $\mathbb{R}^4$.

Solution 1.2.4

1. The family $\{(1,2),(-1,1)\}$ generates $\mathbb{R}^2$ if and only if
   For every $X = (x,y) \in \mathbb{R}^2$, there exist $\lambda, \mu \in \mathbb{R}$ such that $X = \lambda(1,2) + \mu(-1,1)$.
   Let $(x,y) \in \mathbb{R}^2$, let's find $\lambda, \mu \in \mathbb{R}$ such that :
   $(x,y) = \lambda(1,2) + \mu(-1,1) = (\lambda - \mu, 2\lambda + \mu)$
   So we have :
   $$\begin{cases} x = \lambda - \mu \quad (1) \\ y = 2\lambda + \mu \quad (2) \end{cases}$$
   $\Rightarrow \lambda = \frac{x+y}{3}$ and $\mu = \frac{-2x+y}{3}$.
   Hence, this family is generating.
2. Which among the following families are linearly independent :
   $F_1 = \{(1,1,0),(1,0,0),(0,1,1)\}$,
   $F_2 = \{(0,1,1,0),(1,1,1,0),(2,1,1,0)\}$.
   (a) $F_1 = \{(1,1,0),(1,0,0),(0,1,1)\}$ is independent if and only if
   For all λ_1, λ_2, $\lambda_3 \in \mathbb{R}$, $\lambda_1(1,1,0) + \lambda_2(1,0,0) + \lambda_3(0,1,1) = (0,0,0)$
   $$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ \lambda_1 + \lambda_3 = 0 \quad \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0. \\ \lambda_3 = 0 \end{cases}$$
   Therefore, F_1 is linearly independent.
   (b) $F_2 = \{(0,1,1,0),(1,1,1,0),(2,1,1,0)\}$ is not independent because
   There exist $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 1$ in $\mathbb{R}$ such that $\lambda_1(0,1,1,0) + \lambda_2(1,1,1,0) + \lambda_3(2,1,1,0) = (0,0,0,0)$.

3. The family $\{(1,2),(-1,1)\}$ is a basis for $\mathbb{R}^2$, because when the number of vectors $= 2 = \dim \mathbb{R}^2$, it is sufficient to show that it is either generating or independent for it to be a basis. According to question (1), it is generating.
   The family $F_1 = \{(1,1,0),(1,0,0),(0,1,1)\}$ is a basis for $\mathbb{R}^3$, because the cardinality of F_1 is equal to $3 = \dim \mathbb{R}^3$, and F_1 is linearly independent, making it a basis for $\mathbb{R}^3$.