Chapter VI: Numerical Resolution of Ordinary Differential Equations: Cauchy Problem

Differential equations are used in the mathematical modeling of almost all physical phenomena. A differential equation is a relationship between a variable and its derivatives of various orders. In this chapter, we will consider first-order ordinary differential equations with an imposed initial condition (Cauchy problem).

The Cauchy problem is defined by the solution of the following differential equation:

$$\begin{cases} \frac{dy(t)}{dt} = y' = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$ where $y(t_0) = y_0$ is an imposed initial condition.

We note that the solution must necessarily pass through the point (t_0, y_0).

The solution of the Cauchy problem exists and is unique if the function $f(t, y(t))$ satisfies the Lipschitz condition in y on a rectangle R defined by $a \le t \le b$ and $c \le y \le d$. This condition requires that $|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|$, where L is a constant.

In practice, to verify this condition, we calculate $Max\left|\frac{\partial f(t,y)}{\partial y}\right| \le L$ on R.

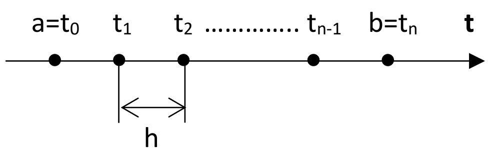
4.1 Euler's Method

Let [a, b] be an interval on which we seek the solution of a Cauchy problem:

$$\begin{cases} \frac{dy(t)}{dt} = y' = f(t, y(t)) \\ y(t_0 = a) = y_0 \end{cases}$$

The function $f(t, y(t))$ satisfies the Lipschitz condition in y on the rectangle R.

The first step is to divide the given interval into n equidistant points, which gives an integration step $h = \frac{b-a}{n}$. The point with abscissa t_i is given by $t_i = a + ih$ (for i = 1, 2, ..., n).



We will therefore solve the problem on the interval $[a = t_0, b = t_n]$ with $y(t_0 = a) = y_0$. If the functions $y(t), y'(t)$ and $y''(t)$ are continuous, we can write the Taylor series expansion for $y(t_1)$ in the neighborhood of t_0, we have :

$$y(t_1) = y(t_0) + y'(t_0)\frac{(t_1 - t_0)}{1!} + y''(t_0)\frac{(t_1 - t_0)^2}{2!} + \cdots$$

Knowing that $y'(t_0) = f(t_0, y(t_0))$ et $h = t_1 - t_0$,

let's write: $y(t_1) = y(t_0) + f(t_0, y(t_0))h + y''(t_0)\frac{h^2}{2!} + \cdots$

If h is sufficiently small, then we can neglect the terms of order two and higher, so we obtain:

$$\boldsymbol{y(t_1) = y(t_0) + hf\big(t_0, y(t_0)\big) \rightarrow y_1 = y_0 + hf(t_0, y_0)}$$

This is the first-order Euler approximation. By repeating the process, we generate a sequence of points $\boldsymbol{y_1, y_2, y_3, \dots\dots, y_{n-1}, y_n}$ approximating $\boldsymbol{y = y(t)}$, in general:

$$\begin{cases} \boldsymbol{t_i = a + ih}, \quad \boldsymbol{i = 0, 1, 2, \dots\dots, n-1} \\ \boldsymbol{y_{i+1} = y_i + hf(t_i, y_i)}, \qquad \boldsymbol{y_0 = y(t_0)} \end{cases}$$

<u>Example:</u>

Solve the following Cauchy problem by Euler's method with an integration step h = 0.25.

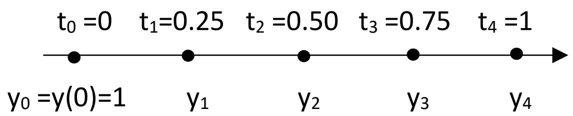
$$\begin{cases} y' = 2 - ty^2 \;\; t \in [0,1], y \in [0,1] \\ \qquad\qquad y(0) = 1 \end{cases}$$

Let's verify the Lipschitz condition on the rectangle R defined by [0, 1] x [0, 1].

$$\text{Max}\left|\frac{\boldsymbol{\partial f(t,y)}}{\boldsymbol{\partial y}}\right| = \text{Max}\left|\frac{\boldsymbol{\partial}(2 - ty^2)}{\boldsymbol{\partial y}}\right| = |\boldsymbol{-2ty}| = \boldsymbol{2 < L}$$

Condition verified.

We divide the interval of t, [0, 1] with a step h = 0.25, i.e., 0, 0.25, 0.50, 0.75, and 1.



We write: $\begin{cases} \qquad\qquad \boldsymbol{t_i = ih}, \quad \boldsymbol{i = 0, 1, 2} \text{ et } \boldsymbol{3} \\ \boldsymbol{y_{i+1} = y_i + hf(t_i, y_i) = y_i + 0.25(2 - t_i y_i^2)}, \;\; \boldsymbol{y_0 = 1} \end{cases}$

$$\boldsymbol{i = 0}, \qquad \boldsymbol{y_1 = y_0 + hf(t_0, y_0) = 1 + 0.25\big(2 - 0 * 1^2\big)} \qquad\qquad \boldsymbol{= 1.5}$$

$$\boldsymbol{i = 1}, \qquad \boldsymbol{y_2 = y_1 + hf(t_1, y_1) = 1.5 + 0.25\big(2 - 0.25 * 1.5^2\big)} \qquad \boldsymbol{= 1.8594}$$

$$\boldsymbol{i = 2}, \qquad \boldsymbol{y_3 = y_2 + hf(t_2, y_2) = 1.859 + 0.25\big(2 - 0.5 * 1.859^2\big)} \quad \boldsymbol{= 1.9272}$$

$$\boldsymbol{i = 3}, \qquad \boldsymbol{y_4 = y_3 + hf(t_3, y_3) = 1.927 + 0.25\big(2 - 0.75 * 1.927^2\big) = 1.7308}$$

The exact solution is:

$$\boldsymbol{y(t) = \frac{2\sqrt[3]{2}c_1 t Ai(2^{2/3}t) + 2\sqrt[3]{2} t Bi(2^{2/3}t)}{2t(c_1 A_i'(2^{2/3}t) + B_i'(2^{2/3}t))}}$$

With Ai and Bi being the Airy functions

and Airy functions of the second kind.

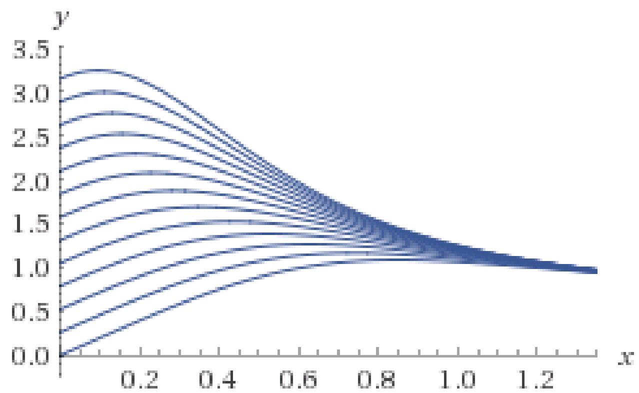


Fig. 4.1. Solutions y(t) for different initial conditions y(0).

4.2 Improved Euler's Method (Heun's Method)

From the Taylor series expansion, we have:

$$y(t_1) = y(t_0) + y'(t_0)\frac{(t_1-t_0)}{1!} + y''(t_0)\frac{(t_1-t_0)^2}{2!} + y'''(t_0)\frac{(t_1-t_0)^3}{3!} + \cdots$$

If we take the first three terms and neglect the others of higher order, we obtain:

$$y(t_1) \cong y(t_0) + y'(t_0)\frac{(t_1-t_0)}{1!} + y''(t_0)\frac{(t_1-t_0)^2}{2!} + \cdots$$

If $h = t_1 - t_0$, this gives:

$$y(t_1) = y(t_0) + y'(t_0)h + y''(t_0)\frac{h^2}{2!}$$

Since $y''(t_0) = \frac{y'(t_1)-y'(t_0)}{h}$, we will have: $y(t_1) = y(t_0) + \frac{h}{2}[f(t_1, y_1) + f(t_0, y_0)]$

In general, the improved Euler formula is written:

$$y_{i+1} = y_i + \frac{h}{2}[f(t_{i+1}, y_{i+1}^E) + f(t_i, y_i)]$$

We note that this formula gives y_{i+1} as a function of y_{i+1}^E, which must be calculated by Euler's method.

Example: Solve the previous Cauchy problem by the improved Euler method with an integration step h = 0.25.

We have:

$$\begin{cases} t_i = ih, \quad i = 0, 1, 2 \text{ et } 3 \\ y_{i+1} = y_i + \frac{h}{2}[f(t_{i+1}, y_{i+1}^E) + f(t_i, y_i)] = y_i + \frac{h}{2}[(2 - t_i y_i^2) + (2 - t_{i+1} y_{i+1}^{2\ E})], \qquad y_0 = 1 \end{cases}$$

With $y_{i+1}^E = y_i + h(2 - t_i y_i^2)$ the solution obtained by the modified or improved Euler method (Heun). We replace y_{i+1}^E, we will have

$$y_{i+1} = y_i + \frac{h}{2}[(2 - t_i y_i^2) + \left(2 - t_{i+1}\left(y_i + h(2 - t_i y_i^2)\right)^2\right)]$$

$y_1 = 1.4297,\ y_2 = 1.6629,\ y_3 = 1.6805,\ y_4 = 1.5750$

4.3 Fourth-Order Runge-Kutta Method

This is the most accurate and widely used method; it is of fourth order. The interval **[a, b]** is divided into **n** subintervals of width h, this formula is written:

$$y_{i+1} = y_i + \frac{h}{6}(K_1 + 2(K_2 + K_3) + K_4)$$

$$K_1 = f(t_i, y_i)$$

$$K_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}K_1\right)$$

$$K_3 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}K_2\right)$$

$$K_4 = f(t_i + h, y_i + hK_3)$$

Example: Solve the previous Cauchy problem by the fourth-order Runge-Kutta method with an integration step h = 0.25.

$$y_{i+1} = y_i + \frac{h}{6}(K_1 + 2(K_2 + K_3) + K_4)$$

$$K_1 = f(t_i, y_i) = 2 - t_i y_i^2$$

$$K_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}K_1\right) = 2 - \left(t_i + \frac{h}{2}\right)\left(y_i + \frac{h}{2}K_1\right)^2$$

$$K_3 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}K_2\right) = 2 - \left(t_i + \frac{h}{2}\right)\left(y_i + \frac{h}{2}K_2\right)^2$$

$$K_4 = f(t_i + h, y_i + hK_3) = 2 - (t_i + h)(y_i + hK_3)^2$$

i=0, $K_1 = 2,\ K_2 = 1.8047,\ K_3 = 1.8122, K_4 = 1.4722, y_1 = 1.4461$

y_2=1.7028, y_3=1.7317, y_4=1.6148

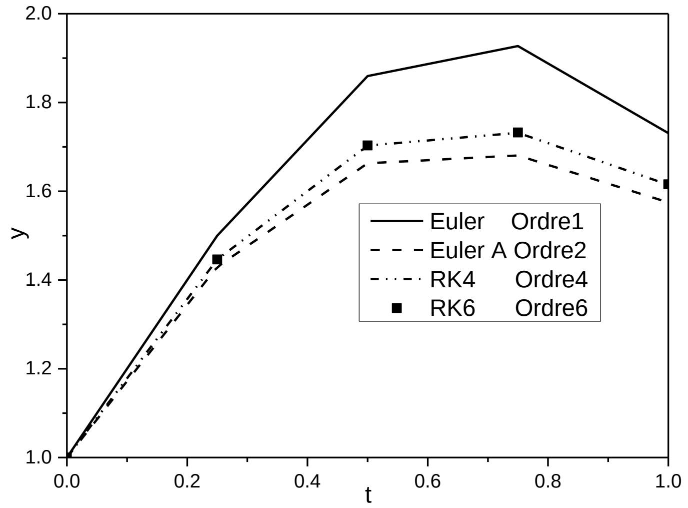


Fig. 4.2. Comparison between different methods