University of Oum El Bouaghi — Academic year: 2024/2025

Analysis 4 — License 2 - Mathematics

Sheet of exercises N°2 - Limits and continuity

Exercise 1 *Find and sketch the domain of each of the following functions:*

1. $f(x,y) = \dfrac{\sqrt{x}y}{x^2+y^2}$
2. $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$
3. $f(x,y) = \dfrac{\sin(xy)}{xy}$
4. $f(x,y) = \ln(xy)$
5. $f(x,y) = \dfrac{\ln(y-x)}{x}$
6. $f(x,y) = \sqrt{1-xy}$
7. $f(x,y) = \dfrac{\sqrt{4-x^2-y^2}}{\sqrt{x^2+y^2-1}}$
8. $f(x,y) = x\ln(y^2-x)$
9. $f(x,y) = \ln(9-x^2-9y^2)$

Exercise 2 *We recall that the* ***level curves*** *of a function f of two variables are the curves with equations $f(x,y) = k$, where k is a constant (in the range of f).*

A level curve $f(x,y) = k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it shows where the graph of f has height k.

1. *Sketch the level curves of the function $f : (x,y) \in \mathbb{R}^2 \mapsto 6-3x-2y$ for the values $k = -6, 0, 6, 12$.*
2. *Sketch the level curves of the function*

$$g : (x,y) \in \mathbb{R}^2 \mapsto \sqrt{9-x^2-y^2} \qquad \text{for} \quad k = 0, 1, 2, 3.$$

Exercise 3 *Find, if they exist, the limits of the following functions at $(0,0)$:*

1. $f : (x,y) \in \mathbb{R}^2 \mapsto \dfrac{x}{\sqrt{x^2+y^2}}$
2. $f : (x,y) \in \mathbb{R}^2 \mapsto \dfrac{|\sin(x+y)|}{\sqrt{x^2+y^2}}$
3. $f : (x,y) \in \mathbb{R}^2 \mapsto \dfrac{\sin(x^2+y^2)}{|x|+|y|}$
4. $f : (x,y) \in \mathbb{R}^2 \mapsto \dfrac{\ln(1+xy)}{\sqrt{x^2+y^2}}$
5. $f : (x,y) \in \mathbb{R}^2 \mapsto \dfrac{xy^2}{x^4+y^3}$
6. $f : (x,y) \in \mathbb{R}^2 \mapsto \dfrac{xy^2}{x^4+y^2}$

Exercise 4 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$f(x,y) = \begin{cases} \dfrac{xy\,\sin(x-y)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Study the continuity of f on $\mathbb{R}^2$.

Exercise 5 *Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) = \begin{cases} 2x^2 + y^2 - 1 & \text{if } x^2 + y^2 > 1, \\ x^2 & \text{else,} \end{cases}$$

is continuous on $\mathbb{R}^2$.

Exercise 6 *Let $\alpha \in \mathbb{R}$. We define the function f_α on $\mathbb{R}^2$ by:*

$$f_\alpha(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^\alpha} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

Using a change of variables in polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$ with $r \in \mathbb{R}_+^*$ and $\theta \in [0, 2\pi[$, study, according to the values of α, the continuity of f_α on $\mathbb{R}^2$.

Exercise 7 *Show that the function defined by $f(x,y) = \dfrac{\sin(xy)}{xy}$ can be extended into a continuous function on $\mathbb{R}^2$.*

Exercise 8 *Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) = \frac{\sin(x^2) - \sin(y^2)}{x^2 + y^2}$$

is not extendable by continuity at $(0,0)$.

Exercise 9

1. *Show that the function $S : \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto x + y$ is Lipschitzian and therefore uniformly continuous.*
2. *Show that any norm on $\mathbb{R}^n$ is a Lipschitz function and therefore uniformly continuous.*
3. *Show that any linear application of $\mathbb{R}^n$ in $\mathbb{R}^p$ is Lipschitzian and therefore uniformly continuous.*