Chapter II: Polynomial Interpolation

For example, consider an experiment where we record the distance traveled by an object as a function of time. The results are given in the following table:

t(sec)	0	1	2	3	4
X(m)	0	5	15	0	3

We want, for example, to calculate the position of the object at time **t=2.5** sec or the speed of the object at a given time. To do this, we need to have an analytical form of **X** as a function of **t**, **X(t)**. This form must at least coincide with the points given in the table. Then we can calculate **X(2.5)**, $\int_0^4 X(t)dt$ or $v(t) = \frac{dX(t)}{dt}$.

In this chapter, we will consider the approximation of **X(t)** by a polynomial form, that is:

$$X(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

where a_i $(i = 0, n)$ are coefficients to be determined.

The polynomials that we will study differ only in the way the coefficients a_i $(i = 0, n)$ are determined, because for a given table of values, the interpolation polynomial is unique.

1 Lagrange Interpolation Polynomial

Let **(n+1)** distinct points $x_0, x_1, \dots \dots, x_n$ and f be a function whose values are $f(x_0), f(x_1), \dots \dots, f(x_n)$. Then, there exists a unique polynomial of degree less than or equal to **n** that coincides with the interpolation points, i.e.:

$$f(x_k) = P_n(x_k), \qquad k = 0, 1, 2, \dots \dots, n$$

This polynomial is given by:

$$P_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + \cdots \dots + f(x_n)L_n(x)$$

where

$$L_k(x) = \sum_{i=0, i\neq k}^{n} \frac{(x-x_i)}{(x_k-x_i)} = \frac{(x-x_0)}{(x_k-x_0)} \frac{(x-x_1)}{(x_k-x_1)} \dots \cdot \frac{(x-x_{k-1})}{(x_k-x_{k-1})} \frac{(x-x_{k+1})}{(x_k-x_{k+1})} \cdots \frac{(x-x_n)}{(x_k-x_n)} \quad (k=0,..,n)$$

$L_k(x)$ are called Lagrange polynomial coefficients. They are orthogonal, that is, $L_k(x_j) = 0$ and $L_k(x_k) = 1$.

Example: Let's take the table given at the beginning of the chapter and try to calculate the Lagrange polynomial for this table. Note that for **n+1** points the degree of the polynomial is less than or equal to **n**. In our case we have **5** points, this gives us a polynomial of degree less than or equal to **4**.

$$X(t) \approx P_4(t) = \sum_{i=0}^{4} f(t_i) L_i(t) = f(t_0)L_0(t) + f(t_1)L_1(t) + f(t_2)L_2(t) + f(t_3)L_3(t) + f(t_4)L_4(t)$$

The coefficients $f(t_i)$ are the values of $\mathbf{X(t_i)}$ at the given points $\mathbf{t_i}$, we substitute and write:

$$X(t) \approx P_4(t) = 0 * L_0(t) + 5 * L_1(t) + 15 * L_2(t) + 0 * L_3(t) + 3 * L_4(t)$$

Then, we calculate the Lagrange polynomial coefficients:

$$L_0(t) = \sum_{i=0,i\neq 0}^{4} \frac{(t-t_i)}{(t_k-t_i)} = \frac{(t-t_1)}{(t_0-t_1)}\frac{(t-t_2)}{(t_0-t_2)}\frac{(t-t_3)}{(t_0-t_3)}\frac{(t-t_4)}{(t_0-t_4)}\ .$$

Note that it is useless to calculate the polynomial coefficients $L_0(t)$ and $L_3(t)$ because they will be multiplied by zero in the substitution.

$$L_1(t) = \sum_{i=0,i\neq 1}^{4} \frac{(t-t_i)}{(t_k-t_i)} = \frac{(t-t_0)}{(t_1-t_0)}\frac{(t-t_2)}{(t_1-t_2)}\frac{(t-t_3)}{(t_1-t_3)}\frac{(t-t_4)}{(t_1-t_4)} = \frac{(t-0)}{(1-0)}\frac{(t-2)}{(1-2)}\frac{(t-3)}{(1-3)}\frac{(t-4)}{(1-4)} = -\frac{1}{6}(t^4 - 9t^3 + 26t^2 - 24t)$$

$$L_2(t) = \sum_{i=0,i\neq 2}^{4} \frac{(t-t_i)}{(t_k-t_i)} = \frac{(t-t_0)}{(t_2-t_0)}\frac{(t-t_1)}{(t_2-t_1)}\frac{(t-t_3)}{(t_2-t_3)}\frac{(t-t_4)}{(t_2-t_4)} = \frac{(t-0)}{(2-0)}\frac{(t-1)}{(2-1)}\frac{(t-3)}{(2-3)}\frac{(t-4)}{(2-4)} = \frac{1}{4}(t^4 - 8t^3 + 19t^2 - 12t)$$

$$L_4(t) = \sum_{i=0,i\neq 4}^{4} \frac{(t-t_i)}{(t_k-t_i)} = \frac{(t-t_0)}{(t_4-t_0)}\frac{(t-t_1)}{(t_4-t_1)}\frac{(t-t_2)}{(t_4-t_2)}\frac{(t-t_3)}{(t_4-t_3)} = \frac{(t-0)}{(4-0)}\frac{(t-1)}{(4-1)}\frac{(t-2)}{(4-2)}\frac{(t-3)}{(4-3)} = \frac{1}{24}(\ t^4 - 6t^3 + 11t^2 - 6t)$$

Finally, we substitute the polynomial coefficients and obtain:

$$X(t) \approx P_4(t) = -25.75t + 50.95833t^2 - 23.25t^3 + 3.04167t^4$$

2. Newton Interpolation Polynomial

We have seen that the Lagrange polynomial uses **(n+1)** polynomial coefficients which are themselves polynomials of degree less than or equal to **n**. The calculation of these polynomial coefficients is also a delicate task, which is why it is interesting to use another, more flexible formulation: the Newton polynomial.

The calculation of the Newton polynomial begins with the construction of a polynomial of degree **1**, $P_1(x)$, which passes through the first two points. Then, the latter will be used to calculate another of degree **2**, $P_2(x)$, which passes through the first three points, and so on until the final polynomial of degree less than or equal to **n**, $P_n(x)$. We have the following recurrence relation between two successive polynomials $P_{i-1}(x)$ and $P_i(x)$ ***(i=2,3,..,n+1)***:

$$\begin{cases} P_1(x) = a_0 + a_1(x-x_0) \\ P_2(x) = P_1(x) + a_2(x-x_0)(x-x_1) \\ P_3(x) = P_2(x) + a_3(x-x_0)(x-x_1)(x-x_2) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. \\ P_n(x) = P_{n-1}(x) + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1}) \end{cases}$$

We note that the coefficients a_k ***(k=0,…,n)*** are the essential elements in the calculation of Newton polynomials. These coefficients are the divided differences of order **k** of the function ***f***.

2.1 Calculation of Divided Differences of a Function f

The divided differences of a function f based on the points $x_0, x_1, \dots\dots, x_n$ are given by:

$$a_k = f[x_0, x_1, \dots\dots, x_k] = \sum_{i=0}^{k} \frac{y_i}{\prod_{j=0,j\neq i}^{k}(x_i - x_j)}$$

In practice, and for a limited number of points, the divided differences are calculated using a table that has the following form:

x_k	$f(x_k)=f[x_k]$	DD^1	DD^2	DD^3	DD^4
x_0	$f[x_0]$				
x_1	$f[x_1]$	$f[x_0,x_1]$	$f[x_0,x_1,x_2]$		
x_2	$f[x_2]$	$f[x_1,x_2]$	$f[x_1,x_2,x_3]$	$f[x_0,x_1,x_2,x_3]$	$f[x_0,x_1,x_2,x_3,x_4]$
x_3	$f[x_3]$	$f[x_2,x_3]$	$f[x_2,x_3,x_4]$	$f[x_1,x_2,x_3,x_4]$	
x_4	$f[x_4]$	$f[x_3,x_4]$			

where

$$f[x_0,x_1]=\frac{f[x_1]-f[x_0]}{x_1-x_0},\quad f[x_1,x_2]=\frac{f[x_2]-f[x_1]}{x_2-x_1},\dots\dots$$
$$f[x_0,x_1,x_2]=\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0},\quad f[x_1,x_2,x_3]=\frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1},\dots\dots$$
$$f[x_0,x_1,x_2,x_3]=\frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0},\quad f[x_1,x_2,x_3,x_4]=\frac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}$$
$$f[x_0,x_1,x_2,x_3,x_4]=\frac{f[x_1,x_2,x_3,x_4]-f[x_0,x_1,x_2,x_3]}{x_4-x_0}$$

3. Interpolation Error $\varepsilon(x)$

This is the error made when replacing the function ***f*** with the equivalent interpolation polynomial. It is denoted by ***ε(x)*** because it varies from one point to another in the interpolation interval. This error must be zero at the interpolation points, $\varepsilon(x_i)=0$, ***(i=0,…n)***.

If the function ***f*** is continuous and ***(n+1)*** times differentiable on the interpolation interval $[a=x_0,\ b=x_n]$, then for all ***x∈[a,b]*** there exists ***z∈[a,b]*** such that:

$$\varepsilon(x)=|f(x)-P_n(x)|=\prod_{i=0}^{n}\frac{(x-x_i)}{(n+1)!}f^{(n+1)}(z)$$

If $\left|f^{(n+1)}(z)\right|\le M\quad\forall z\in[a,b]$, we can write:

$$\varepsilon(x)=|f(x)-P_n(x)|\le\prod_{i=0}^{n}\frac{(x-x_i)}{(n+1)!}M$$

In this case, ***M*** is an upper bound of the function $f^{(n+1)}(x)$ on the interval ***[a,b]***.

Example: Let's take the table given at the beginning of the chapter and try to calculate the Newton polynomial for this table. Note that for n+1 points the degree of the polynomial is less than or equal to n. In our case we have 5 points, this gives us a polynomial of degree less than or equal to 4. Let's write the Newton polynomials:

$$\begin{cases} P_1(t)=a_0+a_1(t-t_0)\\ P_2(t)=P_1(t)+a_2(t-t_0)(t-t_1)\\ P_3(t)=P_2(t)+a_3(t-t_0)(t-t_1)(t-t_2)\\ P_4(t)=P_3(t)+a_4(t-t_0)(t-t_1)(t-t_2)(t-t_3)\\ \text{The } a_i \text{ are the divided differences of order i.}\end{cases}$$

Let's calculate the table of divided differences:

t_k	$f(t_k) = f[t_k]$	DD^1	DD^2	DD^3	DD^4
0	$\mathbf{0 = a_0}$	$\mathbf{5 = a_1}$	$\mathbf{2.5 = a_2}$	$\mathbf{-5 = a_3}$	$\mathbf{3.04167 = a_4}$
1	*5*	10	−12.5	7.1667	
2	*15*	−15	9		
3	*0*	3			
4	*3*				

Substituting the a_i and t_i by their values in the Newton polynomials, we find:

$$\boldsymbol{P_1(t) = a_0 + a_1(t - t_0) = 0 + 5(t - 0) = 5t}$$

$$\boldsymbol{P_2(t) = P_1(t) + a_2(t - t_0)(t - t_1) = 5t + 2.5(t - 0)(t - 1) = 2.5t^2 + 2.5t}$$

$$\boldsymbol{P_3(t) = P_2(t) + a_3(t - t_0)(t - t_1)(t - t_2) = 2.5t^2 + 2.5t - 5(t - 0)(t - 1)(t - 2) = -5t^3 + 17.5t^2 - 7.5t}$$

$$\boldsymbol{P_4(t) = P_3(t) + a_4(t - t_0)(t - t_1)(t - t_2)(t - t_3) = -5t^3 + 17.5t^2 - 6t + 3.0417(t - 0)(t - 1)(t - 2)(t - 3)}$$

$$\boldsymbol{P_4(t)} = 3.04167t^4 - 23.25t^3 + 50.95833t^2 - 25.75t$$

This is the same polynomial as that of Lagrange.

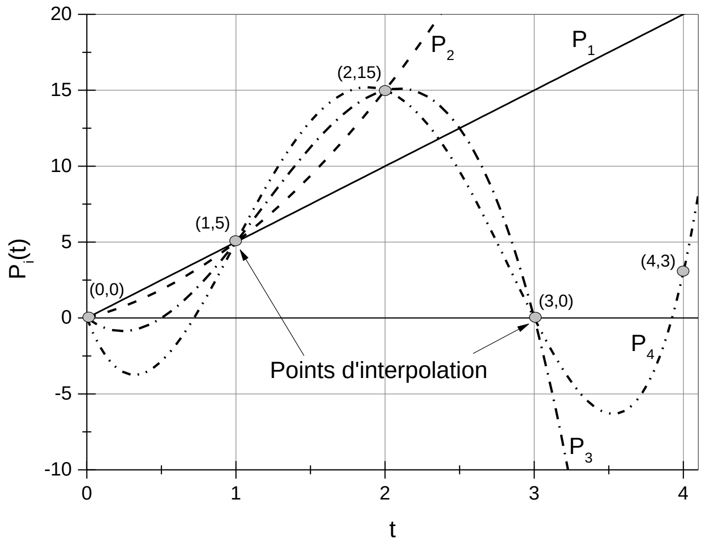

Fig. 2.1. Plots of Newton interpolation polynomials