University of Oum El Bouaghi — Academic year: 2024/2025

Analysis 4 — License 2 - Mathematics

Sheet of exercises N°1

Exercise 1 *Let $\|\cdot\|$ be a norm on a vector space E. Show that for all $x, y \in E$, we have the inequality*

$$|\,\|x\| - \|y\|\,| \leq \|x - y\|.$$

Exercise 2 *Let us recall the usual norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ on $\mathbb{R}^n$:*

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \quad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|, \quad \text{for } x = (x_1, ..., x_n) \in \mathbb{R}^n,$$

and use the notations B_1, B_2 and B_∞ to denote the balls in $\mathbb{R}^n$ corresponding to each of these norms.

1. *Show that for all $x \in \mathbb{R}^n$, we have the inequalities*

   $$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n\,\|x\|_\infty.$$

   What can we conclude from this?

2. *Show that for all $r > 0$ and all $a \in \mathbb{R}^n$, we have the following inclusions:*

   $$B_\infty(a, r/n) \subset B_1(a, r) \subset B_2(a, r) \subset B_\infty(a, r).$$

Exercise 3 *Show that the set $\mathbb{Q}$ of rational numbers is a subset of $\mathbb{R}$ that is neither open nor closed.*

Exercise 4 *Graph the following parts of $\mathbb{R}^2$ and for each of them, say whether it is open, closed, or neither open nor closed.*

$A_1 = \{(x, y) \in \mathbb{R}^2 \ ; \ |x| = 1 \text{ and } |y| \neq 1\}$

$A_2 = \{(x, y) \in \mathbb{R}^2 \ ; \ x + y \geq 0 \text{ and } x > 0\}$

$A_3 = \{(x, y) \in \mathbb{R}^2 \ ; \ x \geq 0 \text{ and } y = 0\}$

$A_4 = \{(x, y) \in \mathbb{R}^2 \ ; \ 1 - xy > 0\}$

$A_5 = \{(x, y) \in \mathbb{R}^2 \ ; \ |x| \neq 1 \text{ and } |y| \neq 1\}$

Exercise 5 *Determine whether each of the following parts of $\mathbb{R}^2$ is bounded or not, compact or not:*

$$A_1 = \left\{(x, y) \in \mathbb{R}^2 \ ; \ x^2 + y^4 = 1\right\}, \qquad A_2 = \left\{(x, y) \in \mathbb{R}^2 \ ; \ x^2 + y^3 = 1\right\}.$$