Limits: A Detailed Lesson for University Students

1. Definition of a Limit

The limit of a function $f(x)$ as x approaches a value a is written as:

$$\lim_{x \to a} f(x) = L$$

This means that as x gets closer and closer to a, the values of $f(x)$ get closer and closer to L, where L is a real number.

2. Intuitive Explanation of Limits

Imagine driving a car towards a stop sign. As you approach the sign, your distance to the sign gets smaller and smaller, but you haven't actually reached the sign yet. The *limit* describes what happens to your distance as you get arbitrarily close to the sign (but not necessarily reaching it).

In the same way, the limit of a function describes what happens to the function's value as x gets arbitrarily close to a particular point a.

3. Formal Definition (Epsilon-Delta Definition)

We say that:

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - a| < \delta$, it follows that $|f(x) - L| < \epsilon$.

In simpler terms, if we can make the function's value as close as we want to L by choosing x sufficiently close to a, then the limit exists and equals L.

4. One-Sided Limits

- **Left-hand limit**: As x approaches a from the left (values less than a):

$$\lim_{x \to a^-} f(x) = L$$

- **Right-hand limit**: As x approaches a from the right (values greater than a):

$$\lim_{x \to a^+} f(x) = L$$

For the **two-sided limit** to exist, both the left-hand and right-hand limits must exist and be equal:

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} = L$$

5. Limits at Infinity

- **Limit as $x \to +\infty$**:

$$\lim_{x \to +\infty} f(x) = L$$

means that as x increases without bound, $f(x)$ gets closer to L.
- **Limit as $x \to -\infty$**:

$$\lim_{x \to -\infty} f(x) = L$$

means that as x decreases without bound, $f(x)$ approaches L.

6. Indeterminate Forms and Limits

Certain limits result in **indeterminate forms** such as: - $\frac{0}{0}$ - $\frac{\infty}{\infty}$ - $0 \cdot \infty$ - $\infty - \infty$ - 1^∞, 0^0, and ∞^0

These forms require special techniques to evaluate, such as **L'Hôpital's Rule**, which can resolve limits of the type $\frac{0}{0}$ and $\frac{\infty}{\infty}$ by differentiating the numerator and denominator:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

7. Common Limit Rules

- **Constant Rule**:
  $$\lim_{x \to a} c = c$$
  for any constant c.
- **Identity Rule**:
  $$\lim_{x \to a} x = a$$

- **Sum Rule**:

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

- **Product Rule**:

$$\lim_{x \to a} [f(x) \cdot g(x)] = \left(\lim_{x \to a} f(x) \right) \cdot \left(\lim_{x \to a} g(x) \right)$$

- **Quotient Rule** (if $\lim_{x \to a} g(x) \neq 0$):

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

- **Power Rule**:

$$\lim_{x \to a} [f(x)]^n = \left(\lim_{x \to a} f(x) \right)^n$$

8. Special Limits

Here are a couple of well-known limits that appear frequently:

- **Sine limit**:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

- **Exponential limit**:

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

9. Techniques for Evaluating Limits

- **Direct substitution**: If $f(x)$ is continuous at a, then:

$$\lim_{x \to a} f(x) = f(a)$$

- **Factoring**: Factor the expression and cancel common terms.
- **Rationalizing**: Multiply by the conjugate when dealing with square roots.
- **L'Hôpital's Rule**: As mentioned earlier, useful for indeterminate forms.

10. Examples of Limits

- **Limit of a Polynomial Function**:

$$\lim_{x \to 2} (3x^2 + 2x - 5) = 3(2)^2 + 2(2) - 5 = 12 + 4 - 5 = 11$$

- **Limit with Indeterminate Form**:

$$\lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0$$

- **Limit as $x \to \infty$ (Rational Function)**:

$$\lim_{x \to \infty} \frac{2x^2 + 3x + 1}{x^2 + 5x - 6} = \frac{2}{1} = 2$$