Lesson: Introduction to Functions

Objective

By the end of this lesson, students will:

- Understand the basic definition of a function.
- Learn about different types of functions.
- Practice identifying domain and range.
- Understand how to evaluate functions for given inputs.

1 What is a Function?

A **function** is a relation between a set of inputs and a set of possible outputs where each input is related to exactly one output. In simpler terms, a function is a "rule" that assigns one output to each input.

Definition

A function f from a set A to a set B is a relation that assigns to each element x in A exactly one element y in B. We write this as:

$$f : A \to B$$

This means f maps each $x \in A$ to exactly one $y \in B$, denoted as $y = f(x)$.

Example

Let's say $f(x) = x^2$. For each input x, the function f gives back $x \times x$ (i.e., the square of x).

$$f(2) = 2^2 = 4$$
$$f(3) = 3^2 = 9$$

Key Terms

- **Domain:** The set of all possible input values (often represented by x).
- **Range:** The set of all possible output values (often represented by y).

For example, if $f(x) = x^2$ and $x \in [0, 5]$, then the domain is $[0, 5]$ and the range is $[0, 25]$.

2 Types of Functions

2.1 Linear Functions

A linear function has the form:

$$f(x) = mx + b$$

where:

- m is the slope of the line (how steep the line is).
- b is the y-intercept (where the line crosses the y-axis).

Example

If $f(x) = 2x + 1$, for each value of x, you can find $f(x)$ by multiplying x by 2 and adding 1.

2.2 Quadratic Functions

A quadratic function is a function of the form:

$$f(x) = ax^2 + bx + c$$

The graph of a quadratic function is a **parabola**.

Example

For $f(x) = x^2 + 3x + 2$, you can calculate $f(x)$ for different values of x to understand the shape of the curve.

2.3 Piecewise Functions

Piecewise functions are defined by different expressions depending on the value of x.

Example

$$f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

2.4 Exponential Functions

An exponential function has the form:

$$f(x) = a \cdot b^x$$

where a is a constant and b is the base.

Example

If $f(x) = 2^x$, this means the function grows exponentially as x increases.

3 Domain and Range

3.1 Domain

The domain of a function is the set of all possible input values that the function can accept.

Example

For the function $f(x) = \frac{1}{x-1}$, the domain excludes $x = 1$, because division by zero is undefined.

3.2 Range

The range of a function is the set of all possible output values.

Example

For the function $f(x) = x^2$, the range is all non-negative numbers because squaring any real number results in a non-negative value.

4 Evaluating Functions

To evaluate a function means to find the output for a specific input.

Example

If $f(x) = 3x - 4$, then:

$$f(2) = 3(2) - 4 = 6 - 4 = 2$$
$$f(0) = 3(0) - 4 = -4$$

5 Graphing Functions

Functions can be represented graphically, and the graph provides a visual understanding of how the function behaves.

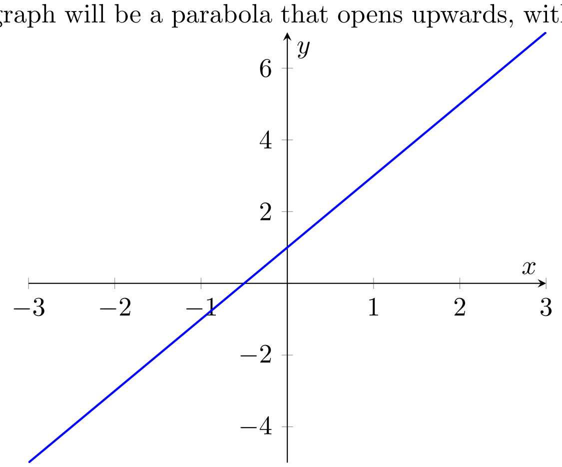
Example: Linear Function

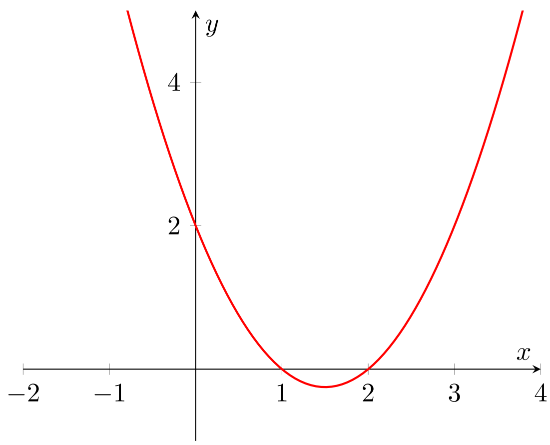
For $f(x) = 2x + 3$, the graph will be a straight line with a slope of 2 and a y-intercept at 3.

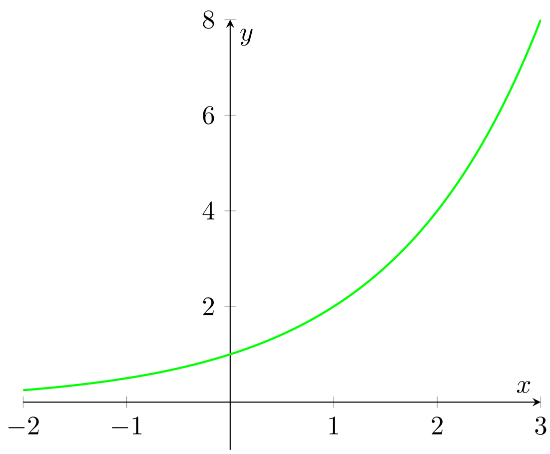
Example: Quadratic Function

For $f(x) = x^2 - 2x + 1$, the graph will be a parabola that opens upwards, with its vertex at the point $(1, 0)$.

Linear function: y = 2x + 1







6 Practice Problems

1. Find the domain and range of the following functions:
   - $f(x) = 2x + 3$
   - $f(x) = \frac{1}{x-5}$
   - $f(x) = \sqrt{x}$
2. Evaluate the following functions for the given values of x:
   - $f(x) = x^2 + 4x + 1$, when $x = 2$ and $x = -1$.
   - $f(x) = \frac{3}{x+1}$, when $x = 2$.
3. Sketch the graph of the function $f(x) = x^2 - 4x + 3$ and determine its vertex and axis of symmetry.

4. Monotonicity

A function $f(x)$ is called:

- **Increasing** if for all x_1, x_2 such that $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$.
- **Decreasing** if for all x_1, x_2 such that $x_1 < x_2$, we have $f(x_1) \geq f(x_2)$.

A function is **strictly increasing** if $f(x_1) < f(x_2)$ for $x_1 < x_2$, and **strictly decreasing** if $f(x_1) > f(x_2)$ for $x_1 < x_2$.

5. Symmetry

A function $f(x)$ is called:

- **Even** if $f(-x) = f(x)$ for all x in the domain. This implies symmetry about the y-axis.
- **Odd** if $f(-x) = -f(x)$ for all x in the domain. This implies symmetry about the origin.

6. Boundedness

A function $f(x)$ is:

- **Bounded above** if there exists a constant M such that $f(x) \leq M$ for all x in the domain.
- **Bounded below** if there exists a constant m such that $f(x) \geq m$ for all x in the domain.
- **Bounded** if it is both bounded above and bounded below, i.e., there exist constants M and m such that $m \leq f(x) \leq M$ for all x in the domain.