Continuous and Discontinuous Functions in Real Analysis

1 Continuous Functions in Real Analysis

In real analysis, a **continuous function** is a type of function that exhibits smooth behavior, meaning there are no abrupt jumps, breaks, or holes in its graph. The formal definition of continuity at a point is crucial for understanding this behavior.

1.1 Formal Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The function f is said to be **continuous at a point** $x_0 \in \mathbb{R}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$:

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

1.2 Intuitive Explanation

This definition essentially tells us that if x is close enough to x_0, then the values of $f(x)$ will be close to $f(x_0)$. In simpler terms, small changes in x around x_0 result in small changes in $f(x)$, without any sudden jumps or discontinuities.

1.3 Continuous on an Interval

A function is said to be **continuous on an interval** $I \subseteq \mathbb{R}$ if it is continuous at every point in I. If the function is continuous on the entire real line $\mathbb{R}$, we simply say that the function is **continuous on** $\mathbb{R}$.

2 Limits and Continuity

The concept of limits is deeply tied to the definition of continuity. A function $f(x)$ is continuous at a point x_0 if the limit of the function as x approaches x_0 exists and is equal to the function value at x_0. This gives rise to the following important relationship:

$$f \text{ is continuous at } x_0 \iff \lim_{x \to x_0} f(x) = f(x_0).$$

In other words, the function $f(x)$ is continuous at x_0 if and only if:

- The limit $\lim_{x \to x_0} f(x)$ exists.
- The function value $f(x_0)$ is defined.
- The limit and the function value are equal, i.e., $\lim_{x \to x_0} f(x) = f(x_0)$.

If any of these conditions fail, the function will not be continuous at x_0.

2.1 One-Sided Limits and Continuity

The limit $\lim_{x \to x_0} f(x)$ can also be approached from the left or the right. A function is continuous at x_0 if both the **left-hand limit** and the **right-hand limit** exist and are equal to each other, as well as to the function value. That is,

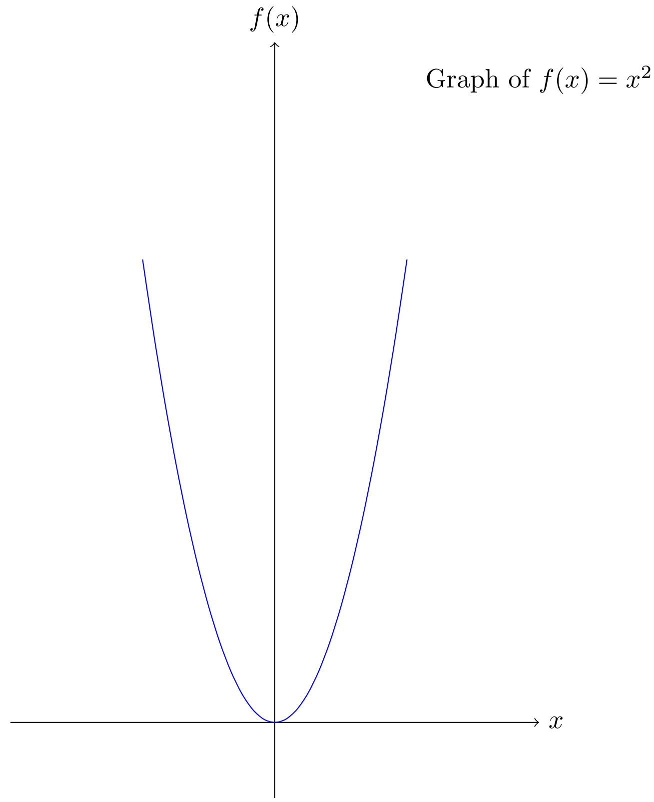
$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0).$$

3 Graphical Representation of Continuous Functions

3.1 Example: Continuous Function $f(x) = x^2$

$$f(x) = x^2$$

The graph of $f(x) = x^2$ is a smooth parabola, and the function is continuous for all $x \in \mathbb{R}$.



Analysis:

The function is continuous at all points $x_0 \in \mathbb{R}$, and at any point x_0, we find that $\lim_{x \to x_0} x^2 = f(x_0)$.

4 Discontinuous Functions

A function is said to be **discontinuous** at a point x_0 if the function fails to be continuous at that point. There are several types of discontinuities, including:

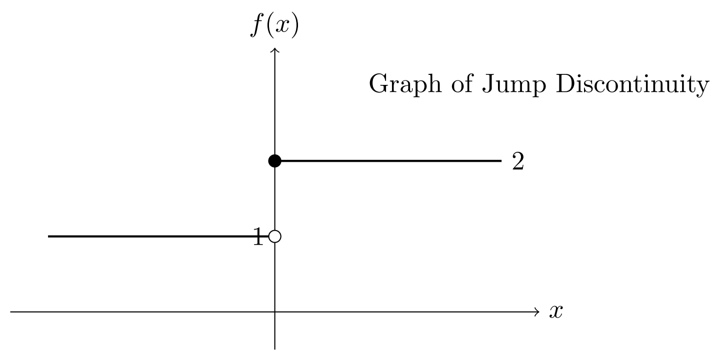
- **Jump Discontinuity**: The function has different left and right limits at x_0.
- **Infinite Discontinuity**: The function approaches infinity near x_0.
- **Removable Discontinuity**: The limit exists but does not equal the function value at x_0.

4.1 Jump Discontinuity

Consider the following function:

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}.$$

The function has a jump discontinuity at $x = 0$, as the left-hand limit and right-hand limit are not equal.



Analysis:

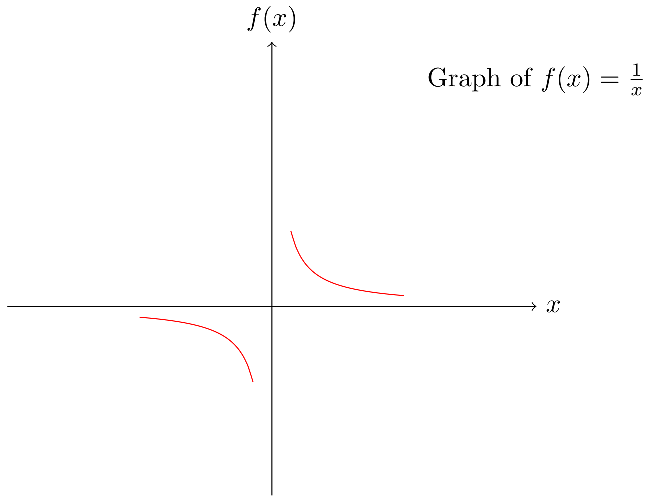
At $x = 0$, the left-hand limit $\lim_{x \to 0^-} f(x) = 1$ and the right-hand limit $\lim_{x \to 0^+} f(x) = 2$ are not equal, so the function is discontinuous at $x = 0$.

4.2 Infinite Discontinuity

Consider the function:

$$f(x) = \frac{1}{x}.$$

The function has an infinite discontinuity at $x = 0$, since as x approaches 0, the values of $f(x)$ grow without bound.



Analysis:

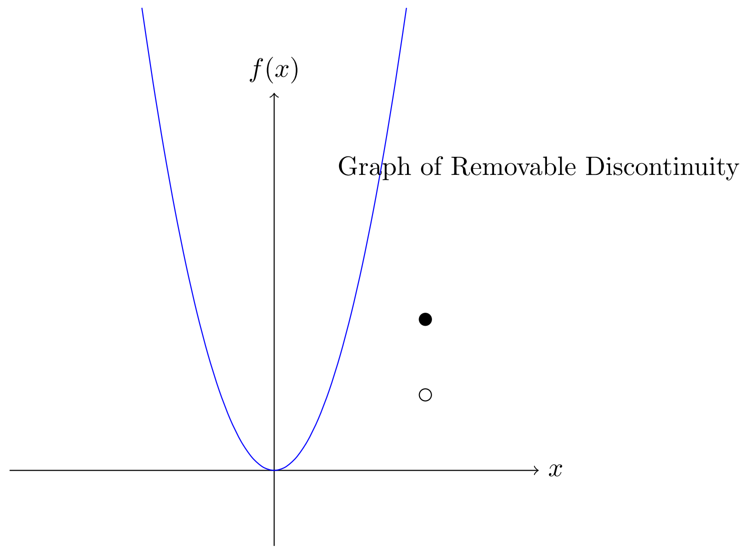
As $x \to 0^+$, $f(x) \to \infty$, and as $x \to 0^-$, $f(x) \to -\infty$. Since $f(x)$ tends toward infinity, the function is discontinuous at $x = 0$.

4.3 Removable Discontinuity

Consider the function:

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}.$$

The function has a removable discontinuity at $x = 1$, since the limit $\lim_{x \to 1} f(x) = 1^2 = 1$, but $f(1) = 2$.



Analysis:

The limit as $x \to 1$ exists and equals 1, but the function value is $f(1) = 2$, which introduces a removable discontinuity at $x = 1$.

5 Key Theorems Related to Continuous Functions

5.1 Intermediate Value Theorem (IVT)

If f is continuous on the closed interval $[a, b]$, then for any value y between $f(a)$ and $f(b)$, there exists at least one $c \in (a, b)$ such that $f(c) = y$. This theorem illustrates the idea that continuous functions take on every value between their endpoints.

Example

Consider $f(x) = x^3$ on the interval $[-1, 1]$. We know that $f(-1) = -1$ and $f(1) = 1$. According to the Intermediate Value Theorem, the function must cross $f(c) = 0$ for some $c \in (-1, 1)$, and indeed, $f(0) = 0$.

5.2 Extreme Value Theorem (EVT)

If f is continuous on a closed and bounded interval $[a, b]$, then f attains both a maximum and a minimum value on $[a, b]$. This means that there are points $x_{\max}, x_{\min} \in [a, b]$ such that $f(x_{\max}) \geq f(x) \geq f(x_{\min})$ for all $x \in [a, b]$.