Let A be

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$$

1) Determine $\ker(A+I)$

2) Deduce that $\lambda = -1$ is an eigenvalue

3) with the help of $\mathrm{tr}(A)$ and $\det A$ determine the other eigenvalues

4) is A diagonable?

1) $\ker(A+I) = \{(x,y,z) \in \mathbb{R}^3, (A+I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\}$ $= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 3 & -1 & 2 \\ 5 & -2 & 3 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3x - y + 2z = 0 \\ 5x - 2y + 3z = 0 \\ -x - z = 0 \end{cases} \Rightarrow \begin{cases} x - y = 0 \\ y = x \\ z = -x \end{cases}$$

$(x,y,z) = x(1,1,-1) \Rightarrow$

$\ker(A+I) = \mathrm{Vect}\{(1,1,-1)\}$

2) AS $\exists v = (1,1,-1) \neq 0$ such that

$Av = -v \Rightarrow (A+I)v = 0 \Rightarrow \lambda = -1$ is an eigen value

3) Let $\lambda_1 = -1, \lambda_2, \lambda_3$ be the eigen values of A, then we have $\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(A) \\ \lambda_1 \lambda_2 \lambda_3 = (-1)^n \det A \end{cases}$

$$\Rightarrow \begin{cases} \lambda_2 + \lambda_3 = -2 \\ \lambda_2 \lambda_3 = 1 \end{cases}$$

so λ_1, λ_2 are zeros of the quadratic equation $X^2 + 2X + 1 = (X+1)^2 \Rightarrow \lambda_2 = \lambda_3 = \lambda_1 = -1$

4) Deduce the char polynomial

4) $C_A(X) = (X-\lambda_1)(\quad)$

$= (X+1)$

5) As we have on the matrix is not

TD : algebra

exercice 1 :

Let A be a square matrix of order 3.
Determine all possible normal jordan forms.
and deduce the characteristic polynomial
and the minimal polynomial of A.

exercice 2 :

Let A be a square matrix of order 3 :
- Determine the characteristic polynomial and the minimal polynomail of A
- Determine all possible normal jordan forms.

solution : 1/ for : $3 = 3+0$

- $3 = 3+0 \longrightarrow P^{-1}AP = J_{\lambda,3} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$
  $= 2+1$
  $= 1+1+1$

$$\Downarrow$$

$$C_A(x) = (x-\lambda)^3 = m_A(x)$$

(because we have only one block which should be of order = the algebraic multiplicity of λ in $m_A(x)$)

For $n : 3 = 2 + 1$.

$P^{-1}AP = J_{\lambda,2} \oplus J_{\lambda,1}$ or $J_{\lambda_1,2} \oplus J_{\lambda_2,1}$

1) $P^{-1}AP = \left(\begin{array}{cc|c} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ \hline 0 & 0 & \lambda \end{array}\right) \Rightarrow C_A(x) = (x-\lambda)^3$

$m_A(x) = (x-\lambda)^2$

2) $P^{-1}AP = \left(\begin{array}{cc|c} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ \hline 0 & 0 & \lambda_2 \end{array}\right) \Rightarrow C_A(x) = (x-\lambda_1)^2(x-\lambda_2)$

$= m_A(x)$

For $n : 3 = 1 + 1 + 1$

1) $P^{-1}AP = J_{\lambda,1} \oplus J_{\lambda,1} \oplus J_{\lambda,1}$

2) $P^{-1}AP = J_{\lambda_1,1} \oplus J_{\lambda_1,1} \oplus J_{\lambda_2,1}$ $(\lambda_1 \neq \lambda_2)$

3) $P^{-1}AP = J_{\lambda_1,1} \oplus J_{\lambda_2,1} \oplus J_{\lambda_3,1}$

$(\lambda_1, \lambda_2, \lambda_3, \lambda_i \neq \lambda_j)$

1/ $P^{-1}AP = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \Rightarrow C_A(x) = (x-\lambda)^3$
$\Rightarrow m_A(x) = (x-\lambda)$

2/ $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \Rightarrow C_A(x) = (x-\lambda_1)^2(x-\lambda_2)$
$\Rightarrow m_A(x) = (x-\lambda_1)(x-\lambda_2)$

3/ $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \Rightarrow C_A(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$
$\Rightarrow m_A(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$
$C_A(x) = m_A(x)$

solution 2:

since A is of order 3. Then the characteristic polynomial is of degree 3. Therefore there exists at least a root λ_1 as eigenvalue of A.

Let $P(x) = x^2 + ax + b$ be $C_A(x) / (x-\lambda_1)$

1) if $P(x)$ has no roots in $\mathbb{R}$, Then there is no Jordan form on $\mathbb{R}$ and $m_A(x) = C_A(x) = (x-\lambda_1).P(x)$ consequently its normal Jordan form on $\mathbb{C}$ is:

3) $P_A^{-1}P = J_{\lambda_1,1} \oplus J_{\lambda_2,1} \oplus J_{\lambda_3,1}$

and $C_A(x) = (x-\lambda_1)(x-\lambda_2)(x-\overline{\lambda_2}) = m_A(x)$

2/ if $P(x)$ has double root :

$P(x) = (x-\lambda_2)^2$. Then :

$C_A(x) = (x-\lambda_1)(x-\lambda_2)^2$

$m_A(x) = \begin{cases} (x-\lambda_1)(x-\lambda_2) \quad \text{--- (1)} \\ C_A(x) = (x-\lambda_1)(x-\lambda_2)^2 \quad \text{--- (2)} \end{cases}$

(1) : $P^{-1}AP = J_{\lambda_1,1} \oplus J_{\lambda_2,1} \oplus J_{\lambda_2,1}$

(2) : $P^{-1}AP = J_{\lambda_1,1} \oplus J_{\lambda_2,2}$

3) if $P(x)$ has two roots λ_2, λ_3

Then $C_A(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3) = m_A(x)$

$P^{-1}AP = J_{\lambda_1,1} \oplus J_{\lambda_2,1} \oplus J_{\lambda_3,1}$

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$P^{-1}AP = J_{\lambda_1,1} \oplus J_{\lambda_2,1} \oplus J_{\lambda_3,1}$

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$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

1) $C_A(x) = \det(xI - A) =$

$$\begin{vmatrix} x & -1 & 0 \\ 4 & x-4 & 0 \\ 2 & -1 & x-2 \end{vmatrix}$$

$= x(x-4)(x-2) + 4(x-2)$

$= (x-2)(x^2 - 4x + 4) = (x-2)^3$

2) $\lambda_1 = \lambda_2 = \lambda_3 = 2$

the eigenVectors :

$$(A - 2I)\begin{pmatrix} \alpha \\ y \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -2\alpha + y = 0 \\ -4\alpha + 2z = 0 \\ -2\alpha + y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} y = 2\alpha \\ z \in \mathbb{R}. \end{cases}$$

$\Rightarrow (\alpha, y, z) = (\alpha, 2\alpha, z)$

$= \alpha(1, 2, 0) + z(0, 0, 1)$.

Since We halle only two Vectors. Then A is not diagonalle

3) $\ker (A - 2I)^2 =$

$\{(\alpha, y, z) \in \mathbb{R}^3, (A - 2I)^2 \begin{pmatrix} \alpha \\ y \\ z \end{pmatrix} = 0\}$

$$(A - 2I)^2 = \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \forall n \geq 2 ; (A - 2I)^n = 0$

Methode 1 :

$$0 = (A - 2I)^n = \sum_{k=0}^{n} C_n^k (-2)^k A^{n-k}$$

$= A^n - 2n A^{n-1}$

$$\Rightarrow A^n = -\sum_{k=0}^{n-1} C_{n-1}^{k} (-2)^k A^{(n-1)-k}$$

Methode 2 :

Since $(A - 2I)^2 = 0$

and $A - 2I \neq 0 \Rightarrow m_A(x) = (x-2)^2$

* the normal jordan form is :

$$P^{-1}AP = J_{2,3} \oplus J_{2,2} \oplus J_{2,1}$$

$$= \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)$$

$$P^{-1}A^nP = (J_{2,2})^n \oplus (J_{2,1})^n$$

$$J_{2,2}^n = (2I + N)^n$$

$$(2I + N)^n = \sum_{k=0}^{n} C_n^k N^k 2^{n-k}$$

$= 2^n I + 2^{n-1} N$

$$= \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix} + 2^{n-1} n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2^n & 2^{n-1} n \\ 0 & 2^n \end{pmatrix}$$

then $A^n = \begin{pmatrix} 2^n & 2^{n-1} n & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}$