Correction de la Serie N° 4

Exo1

1) On sait que $\Phi_X(t) = E(e^{itX})$ et $\Phi_X(0) = 1$.

Donc $\Phi_X(0) = ce = 1 \Leftrightarrow c = e^{-1}$ et $\Phi_X(t) = e^{-4|t|}$ $t \in \mathbb{R}$.

2) Par definition $\Phi_X(t) = \int_{\mathbb{R}} e^{itx} f_X(x)\,dx$ donc

$$f_X(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\Phi_X(t)\,dt \quad \text{(La transformation inverse de Fourier)}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-4|t|-itx}\,dt = \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{(4-ix)t} + \int_{0}^{+\infty} e^{-(4+ix)t}\,dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{4-ix} e^{(4-ix)t}\Big|_{-\infty}^{0} - \frac{1}{4+ix} e^{-(4+ix)t}\Big|_{0}^{+\infty}\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{4-ix} + \frac{1}{4+ix}\right) = \frac{1}{2\pi}\left(\frac{4+ix+4-ix}{16+x^2}\right)$$

$$= \frac{1}{2\pi}\,\frac{8}{16+x^2} = \frac{4}{\pi(16+x^2)}. \quad x \in \mathbb{R}.$$

Exo2 :

On a $\varphi_{X+Y}(t) = E\left(e^{it(X+Y)}\right) = E\left(e^{itX} e^{itY}\right)$

$$\overset{X,Y \text{ indép}}{=} E(e^{itX})E(e^{itY}) = \varphi_X(t)\varphi_Y(t)$$

On a $f_X(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$ si $x \in \mathbb{R}$.

$$\varphi_X(x) = E(e^{itX}) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{itx - \frac{(x-\mu_X)^2}{2\sigma_X^2}}\,dx.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{\mathbb{R}} \exp\left[\frac{-1}{2\sigma_x^2}\left\{ x^2 - 2\mu_x x + \mu_x^2 - 2i\sigma_x^2 t x \right\}\right] dx.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{\mathbb{R}} \exp \frac{-1}{2\sigma_x 2}\Big[x^2 - 2(\mu_x + i\sigma_x^2 t)x + \mu_x^2 + 2i\mu_x\sigma_x^2 t + (i\sigma_x^2 t)^2$$

$$- 2i\mu_x\sigma_x^2 t - (i\sigma_x^2 t)^2\Big] \, dx.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} \int_{\mathbb{R}} \exp \frac{-1}{2\sigma_x^2} \left(x - \mu_x - \underline{i\sigma_x^2 t}\right)^2 e^{i\mu t - \frac{t}{2}\sigma_x^2} \, dx$$

$$\varphi_X(t) = e^{i\mu_x t - \frac{\sigma_x^2}{2} t}.$$

donc $\varphi_Y(t) = e^{i\mu_Y t - \frac{\sigma_Y^2}{2} t}$

2) $\varphi_{X+Y}(t) = \varphi_X(t)\,\varphi_Y(t) = e^{i\mu_x t - \frac{\sigma_x^2}{2} t} \, e^{i\mu_Y t - \frac{\sigma_Y^2 t}{2}}$

$$= e^{i(\mu_x + \mu_Y)t} \, e^{-\frac{(\sigma_x^2 + \sigma_Y^2)}{2} t}$$

Alors $X+Y \sim \mathcal{N}(\mu_x + \mu_Y, \sigma_x^2 + \sigma_Y^2)$.

Exo 3 :

1) on a $P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$ $k \in \mathbb{N}$

$$\varphi_X(t) = E(e^{itX}) = \sum_{k\in\mathbb{N}} e^{itk} P(X=k) = \sum_{k\in\mathbb{N}} e^{itk} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k\in\mathbb{N}} \frac{(\lambda e^{it})^k}{k!} = e^{\lambda(e^{it} - 1)}.$$

2) $\varphi_Z(t) = \varphi_{\sum_{i=1}^{n} X_i}(t) = \prod_{i=1}^{n} \varphi_{X_i}(t)$

$= \prod_{i=1}^{n} e^{n\lambda(e^{it}-1)}$

Alors $Z \sim \mathcal{P}(n\lambda)$.