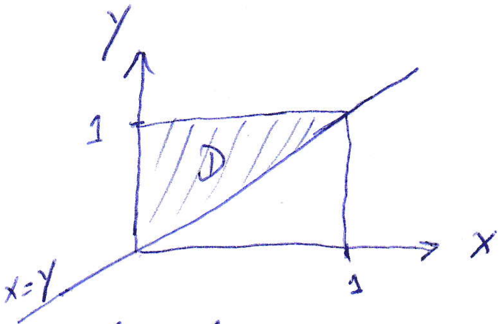
# - Correction interrogation II -

**Exo1** 4,75

1) 



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2) $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1 \Leftrightarrow k\int_0^1 \left[\int_x^1 y^{-1/2}\,dy\right] x^{-1/2}\,dx = 1$

$\Leftrightarrow k\int_0^1 \left[\frac{1}{-1/2+1} y^{-1/2+1}\right]_x^1 x^{-1/2}\,dx = 1$

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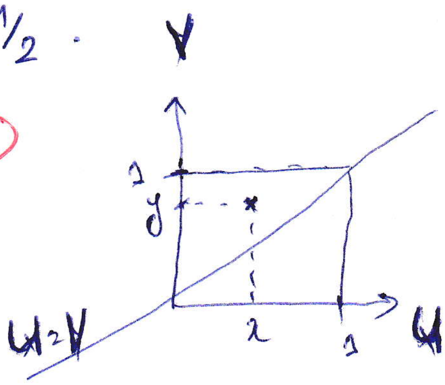
$\Leftrightarrow k\int_0^1 2(1-\sqrt{x})\,x^{-1/2}\,dx = 1$

$\Leftrightarrow 2k\int_0^1 (x^{-1/2}-1)\,dx = 2k\left[2\sqrt{x}-x\right]_0^1 = 1$

$\Leftrightarrow 2k = 1 \Leftrightarrow k = 1/2.$

3) si $x$ ou $y < 0$ $\quad F_{X,Y}(x,y) = 0.$ 0,25



* si $0 \leq x \leq y < 1.$

$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$

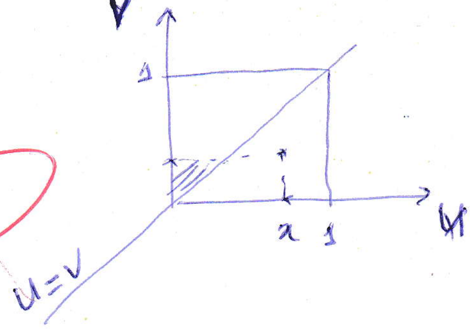
$= \frac{1}{2}\int_0^x\left[\int_u^y f(u,v)\,dv\right]du = \frac{1}{2}\int_0^x\left[\int_u^y v^{-1/2}\,dv\right]u^{-1/2}\,du$ 0,25

$= \frac{1}{2}\int_0^x \left[2\sqrt{v}\right]_u^y u^{-1/2}\,du = \int_0^x (\sqrt{y}-\sqrt{u})\,u^{-1/2}\,du$

$= \int_0^x (\sqrt{y}\,u^{-1/2}-1)\,du = \left[\sqrt{y}\,2\sqrt{u}-u\right]_0^x = 2\sqrt{xy}-x.$

* Si $0 \leq y \leq x$ et $y \leq 1$

$$F_{X,Y}(x,y) = \int_0^y \left[ \int_0^v f(u,v)\,du \right] dv$$

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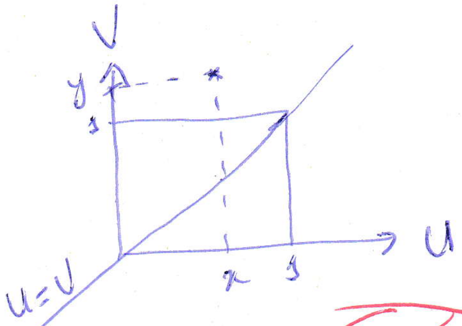
$$= \frac{1}{2} \int_0^y \left[ \int_0^v u^{-1/2}\,du \right] (v)^{-1/2}\,dv$$

$$= \frac{1}{2} \int_0^y \left[ 2\sqrt{u} \right]_0^v v^{-1/2}\,dv = \frac{1}{2} \int_0^y 2\sqrt{v}\, v^{-1/2}\,dv = y.$$




* Si $0 \leq x \leq 1 < y$

$$F_{XY}(x,y) = \int_0^x \left[ \int_u^1 f_{UV}(u,v)\,dv \right] du$$

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$$= \frac{1}{2} \int_0^x \left[ \int_u^1 \frac{1}{\sqrt{v}}\,dv \right] \frac{1}{\sqrt{u}}\,du = \frac{1}{2} \int_0^x \left[ 2\sqrt{v} \right]_u^1 \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{2} \int_0^x 2\left(1 - \sqrt{u}\right) \frac{1}{\sqrt{u}}\,du = \int_0^x \left( \frac{1}{\sqrt{u}} - 1 \right) du = \left[ 2\sqrt{u} - u \right]_0^x$$

$$F_{XY}(x,y) = 2\sqrt{x} - x$$

* Si $x \geq 1$ ou $y \geq 1$.

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$$F_{X,Y}(x,y) = 1$$

Alors

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{si } x \text{ ou } y < 0 \\ 2\sqrt{xy} - x & \text{si } 0 \leq x \leq y < 1 \\ y & \text{si } 0 \leq y \leq x \text{ et } y \leq 1 \\ 2\sqrt{x} - x & \text{si } 0 \leq x \leq 1 < y \\ 1 & \text{si } x \text{ et } y \geq 1 \end{cases}$$

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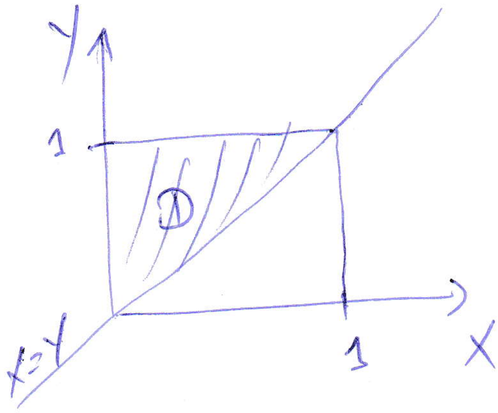
3) $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$




$= \frac{1}{2\sqrt{x}} \int_x^1 \frac{1}{\sqrt{y}}\, dy$

ok

$= \frac{1}{\sqrt{x}} \left[\sqrt{y}\right]_x^1 = \frac{1}{\sqrt{x}}\left(1 - \sqrt{x}\right) = \frac{1}{\sqrt{x}} - 1 \qquad 0 \leq x < 1$

$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx = \frac{1}{2\sqrt{y}} \int_0^y \frac{1}{\sqrt{x}}\, dx = \frac{1}{\sqrt{y}}\left[2\sqrt{x}\right]_0^y = 1 \quad 0 \leq y < 1.$

ok

$f_X(x) \times f_Y(y) = \left(\frac{1}{\sqrt{x}} - 1\right) \times 1 = \frac{1}{\sqrt{x}} - 1 \neq \frac{1}{2\sqrt{xy}} = f_{X,Y}(x,y)$

ok

donc X, Y ne sont pas indépendantes

4) $f_{X/Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{2\sqrt{xy}}}{1} = \frac{1}{2\sqrt{xy}}$ si $0 \leq x \leq y < 1$

ok

$f_{Y/X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{2\sqrt{xy}}}{\frac{\sqrt{x} - 1}{\sqrt{x}}} = \frac{1}{2(1-\sqrt{x})\sqrt{y}} \quad 0 \leq x \leq y < 1.$

ok

5) $E(Y/X=x) = \int_{-\infty}^{+\infty} y f_{Y/X=x}(y)\, dy = \frac{1}{2(1-\sqrt{x})} \int_x^1 \frac{y}{\sqrt{y}}\, dy$

$= \frac{1}{2(1-\sqrt{x})} \int_x^1 \sqrt{y}\, dy = \frac{1}{2(1-\sqrt{x})} \left[\frac{2}{3} y\sqrt{y}\right]_x^1$

$\frac{1 - x\sqrt{x}}{3(1-\sqrt{x})} = \frac{1 + x - \cancel{x} - x\sqrt{x}}{3(1-\sqrt{x})} = \frac{1}{3}\left(1 + x + \sqrt{x}\right).$

car $\dfrac{1-x\sqrt{x}}{3\,(1-\sqrt{x})} = \dfrac{1}{3}\,\dfrac{1-x+x-x\sqrt{x}}{1-\sqrt{x}}$

$= \dfrac{1}{3}\left[\dfrac{1-x}{1-\sqrt{x}} + x\,\dfrac{1-\sqrt{x}}{1-\sqrt{x}}\right] = \dfrac{1}{3}\left\{\left[\dfrac{1-\sqrt{x}+\sqrt{x}-x}{1-\sqrt{x}}\right]+x\right\}$

$= \dfrac{1}{3}\left[\dfrac{1-\sqrt{x}}{1-\sqrt{x}} + \sqrt{x}\,\dfrac{1-\sqrt{x}}{1-\sqrt{x}} + x\right]$

$= \dfrac{1}{3}\left(1+\sqrt{x}+x\right)$

* On a $E(Y/X)$ est une v.a comme la v.a $X$

$$E\left[E(Y/X)\right] = \int_{-\infty}^{+\infty} E(Y/X)\, f_X(x)\,dx$$

$$= \frac{1}{3}\int_0^1 \left(1+\sqrt{x}+x\right)\left(\frac{1}{\sqrt{x}}-1\right)dx$$

$$= \frac{1}{3}\int_0^1 \left(\frac{1}{\sqrt{x}} + \cancel{1} + \cancel{\sqrt{x}} - \cancel{1} - \cancel{\sqrt{x}} - x\right)dx$$

$$= \frac{1}{3}\int_0^1 \left(\frac{1}{\sqrt{x}} - x\right)dx = \frac{1}{3}\left[2\sqrt{x} - \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{3}\times\frac{3}{2} = \boxed{\frac{1}{2}}.$$

D'autre côté $E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 y\,dy = \left.\dfrac{y^2}{2}\right]_0^1 = 1/2$.

Exo 2 1,25

$X, Y \sim U_{[-1,1]}$ donc

$$f_X(x) = \begin{cases} 1/2 & x\in[-1,1] \\ 0 & \text{si non.} \end{cases}$$

$|\!\!\times\!\!|$

On pose $Z = Y - X \Rightarrow Y = Z + X$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x)\, f_Y(z+x)\, dx \quad \text{---} (1)$$

0,25

$x, y \in ]-1, 1[ \Rightarrow z \in ]-2, 2[$ car $-1 \leq -x \leq 1$, $-1 \leq y \leq 1$

* On a $f_Y(z+x) = \begin{cases} \frac{1}{2} & z+x \in ]-1,1[ \iff -1-z < x < 1-z \\ 0 & \text{sinon} \end{cases}$

l'intégral (1) est non nulle pour $-1 < x < 1$ et $-1-z < x < 1-z$

c-à-d pour $\max\{-1, -1-z\} < x < \min\{1, 1-z\}$

* Si $-2 \leq z < 0$ on a $-1 < -1-z < x < 1 < 1-z$

$$f_Z(z) = \int_{-1-z}^{1} f_X(x) f_Y(z+x)\, dx = \int_{-1-z}^{1} \frac{1}{4}\, dx$$

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$$= \frac{1}{4}\left[x\right]_{-1-z}^{1} = \frac{1}{4}(1+1+z) = \frac{1}{4}(2+z)$$

* Si $2 \geq z \geq 0$ $\quad -1-z < -1 < x < 1-z < 1$.

$$f_Z(z) = \int_{-1}^{1-z} f_X(x) f_Y(z+x)\, dx = \int_{-1}^{1-z} \frac{1}{4}\, dx$$

0,5

$$= \frac{1}{4}\left[x\right]_{-1}^{1-z} = \frac{1}{4}(1-z+1) = \frac{1}{4}(2-z).$$

Alors

$$f_Z(z) = \begin{cases} \frac{1}{4}(2-z) & 0 \leq z < 2 \\ \frac{1}{4}(2+z) & -2 \leq z \leq 0 \\ 0 & \text{sinon} \end{cases}$$

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