University of Oum El Bouaghi — Academic year: 2024/2025
Introduction to topology — License 2 - Mathematics

Sheet of exercises N°2

Exercise 1 *Let* $X =]0, +\infty[$. *For* $x, y \in X$, *we set*

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|.$$

1. *Show that* d *is a distance on* X.
2. *Show that, in the metric space* (X, d), $\mathbb{N}^*$ *is a bounded subset and* $\left\{\frac{1}{n}\ ;\ n \in \mathbb{N}^*\right\}$ *is an unbounded subset.*
3. *Is the metric space* (X, d) *complete? (Indication. Consider the real sequence* $x_n = n$*).*

Exercise 2 (Construction of distances) *We call a gauge a map* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *which is non-decreasing, only vanishing at* 0 *and sub-additive (i.e.* $\varphi(s+t) \leq \varphi(s)+\varphi(t)$, $\forall s, t \in \mathbb{R}_+$*).*

1. *Let* (X, d) *be a metric space and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *a gauge. Show that* $d' = \varphi \circ d$ *is a distance on* X.
2. *Application: Deduce that*

$$d_1 = \frac{d}{1+d}, \qquad d_2 = \min\{1, d\}, \qquad d_3 = \ln(1+d), \qquad \text{and} \qquad d_4 = d^\alpha \text{ with } 0 < \alpha < 1$$

are distances on X.

3. *Show that* d *and* $d' = \varphi \circ d$ *are metrically equivalent if there exists a real constant* $C > 0$ *such that*

$$C^{-1}t \leq \varphi(t) \leq Ct, \qquad \forall t \in \mathbb{R}^+.$$

Exercise 3 *Let* (X, d) *be a metric space and let* A *and* B *be two non-empty subsets of* X. *Show that:*

1. $\forall x, y \in X : \quad |d(x, A) - d(y, A)| \leq d(x, y)$,
2. $\forall x \in X : \quad x \in \overline{A} \iff d(x, A) = 0$,
3. $\operatorname{diam}(\overline{A}) = \operatorname{diam}(A)$.

Exercise 4 *Let* (X, d) *be a metric space. Show that:*

1. *Any intersection of complete parts of* (X, d) *is complete.*
2. *A finite union of complete subsets of* (X, d) *is complete.*

3. *If all closed and bounded subsets are complete, then (X, d) is complete.*

Exercise 5 *Let d and d' be two metrically equivalent distances on a set X. Show that:*

1. *(X, d) and (X, d') have the same bounded parts.*
2. *(X, d) and (X, d') have the same convergent sequences and the same Cauchy sequences.*
3. *(X, d) is complete if and only if (X, d') is complete.*

Exercise 6 (Distance of the uniform convergence) *Let X be any non-empty set and let (Y, d) be a metric space. We denote by $\mathcal{B}(X, Y)$ the set of all maps $f : X \to Y$ which are bounded, that is to say which verify: $\operatorname{diam}[f(X)] < +\infty$.*

For $f, g \in \mathcal{B}(X, Y)$, we denote

$$d_\infty(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

1. *Show that d_∞ is a distance on $\mathcal{B}(X, Y)$.*
2. *Show that $(\mathcal{B}(X, Y), d_\infty)$ is a complete space if (Y, d) is complete.*

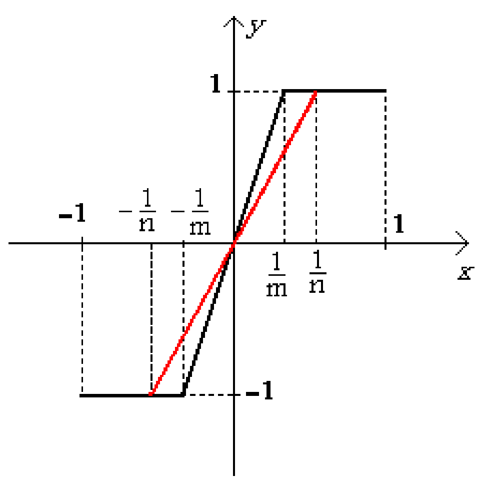
Exercise 7 *Show that for all bounded interval $[a, b]$ of $\mathbb{R}$, the space $C([a, b], \mathbb{R})$, of the continuous functions on $[a, b]$ with values in $\mathbb{R}$, is a complete space for the distance of the uniform convergence d_∞.*

Exercise 8 *Let $C([-1, 1], \mathbb{R})$ be the set of continuous functions $f : [-1, 1] \to \mathbb{R}$. We equip $C([-1, 1], \mathbb{R})$ with the distance d_1 defined by*

$$d_1(f, g) = \int_{-1}^{1} |f(x) - g(x)| \, dx.$$

We will show that $C([-1, 1], \mathbb{R})$ equipped with this distance is not complete. For this, we consider the sequence $(f_n)_{n \geq 1}$ of functions defined by

$$f_n(x) = \begin{cases} -1 & \text{for } x \in [-1, -1/n], \\ nx & \text{for } x \in [-1/n, 1/n], \\ 1 & \text{for } x \in [1/n, 1]. \end{cases}$$



1. *Verify that* $f_n \in C([-1,1],\mathbb{R})$ *for all* $n \geq 1$.

2. *Show that for all* $n, m \geq 1$ *:*

$$|f_m(x) - f_n(x)| \leq 2, \ \forall x \in [-1,1].$$

Deduce that $(f_n)_{n\geq 1}$ *is a Cauchy sequence.*

3. *Suppose there exists a function* $f \in C([-1,1],\mathbb{R})$ *such that* $(f_n)_{n\geq 1}$ *converges to* f *in* $(C([-1,1],\mathbb{R}), d_1)$. *Show that we then have*

$$\lim_{n\to+\infty} \int_{-1}^{-\alpha} |f_n(x) - f(x)|\, dx = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \int_{\alpha}^{1} |f_n(x) - f(x)|\, dx = 0$$

for all $0 < \alpha < 1$.

4. *Show that*

$$\lim_{n\to+\infty} \int_{-1}^{-\alpha} |f_n(x) + 1|\, dx = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \int_{\alpha}^{1} |f_n(x) - 1|\, dx = 0$$

for all $0 < \alpha < 1$.

5. *Deduce that*

$$f(x) = \begin{cases} -1, & x \in [-1, 0[, \\ 1, & x \in]0, 1]. \end{cases}$$

Conclude.

Exercise 9 *Let* (A, d_A) *be a metric subspace of* (X, d). *Show that the topology associated with* d_A *is the topology induced on* A *by the topology on* X *associated with* d.

Exercise 10 (Fixed point theorem and system resolution) *We equip* $\mathbb{R}^2$ *with the distance* d_1 :

$$d_1\left((x,y),(x',y')\right) = |x - x'| + |y - y'|.$$

and we define the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ *by:*

$$f(x,y) = \left(\frac{1}{4}\sin(x+y), 1 + \frac{2}{3}\arctan(x-y)\right).$$

1. *Show that there exists a constant* $k \in]0,1[$ *such that, whatever* (x,y), $(x',y') \in \mathbb{R}^2$, *we have*

$$d_1\left(f(x,y), f(x',y')\right) \leq k \ d_1\left((x,y),(x',y')\right).$$

2. *Deduce that the system*

$$\begin{aligned} \tfrac{1}{4}\sin(x+y) &= x \\ 1 + \tfrac{2}{3}\arctan(x-y) &= y \end{aligned} \qquad (S)$$

admits a unique solution in $\mathbb{R}^2$.

Theorem 1 (Banach-Picard contraction fixed point theorem) *Let* (X, d) *be a complete metric space. If the map* $f : X \to X$ *is a contraction with ratio* k*, then it admits a unique fixed point* $x^* \in X,\ f(x^*) = x^*$*.*
Moreover, any recurrent sequence given by

$$\begin{cases} x_{n+1} = f(x_n), \\ x_0 \in X, \end{cases}$$

converges to x^**, and we have*

$$\forall n \in \mathbb{N} : \quad d(x_n, x^*) \leq k^n\ d(x_0, x^*) \qquad \textit{and} \qquad d(x_n, x^*) \leq \frac{k^n}{1-k}\ d(x_0, x_1).$$