University of Oum el Bouaghi 2024/2025

Faculty of SESNV

Department of Biology

Module : Mathematics-Statistics

Serie Tutorial N^{0}1

Exercise 0.1 *Domain of definition*

$f_1(x) = \dfrac{1}{4-x^2}$, $f_2(x) = \dfrac{1}{\sqrt{4-x^2}}$, $f_3(x) = \sqrt{x-x^3}$, $f_4(x) = \sqrt[3]{x+1}$, $f_5(x) = \ln\left(\dfrac{2+x}{2-x}\right)$,
$f_6(x) = \sqrt{\dfrac{x^2-2}{(x-1)(x+1)}}$, $f_7(x) = \dfrac{\cos x}{e^x-1}$, $f_8(x) = \sqrt{\ln(x)+1}$

f_1 *defined* $\Longleftrightarrow 4-x^2 \neq 0$, *then* $D_{f_1} = \mathbb{R} - \{-2, 2\}$
f_2 *defined* $\Longleftrightarrow 4-x^2 \succ 0$, *then* $D_{f_2} =]-2, +2[$
f_3 *defined* $\Longleftrightarrow x-x^3 \geqslant 0$, *then* $D_{f_3} =]-\infty, -1] \cup [0, +1]$
$D_{f_4} = \mathbb{R}$
f_5 *defined* $\Longleftrightarrow \dfrac{2+x}{2-x} \succ 0$ *and* $2-x \neq 0$, *then* $D_{f_5} =]-2, +2[$
f_6 *defined* $\Longleftrightarrow \dfrac{x^2-2}{(x-1)(x+1)} \geqslant 0$ *and* $(x-1)(x+1) \neq 0$, *then* $D_{f_6} = \left]-\infty, -\sqrt{2}\right] \cup]-1, +1[\cup \left]+\sqrt{2}, +\infty\right[$
f_7 *defined* $\Longleftrightarrow e^x - 1 \neq 0$, *then* $D_{f_7} = \mathbb{R}^*$
f_8 *defined* $\Longleftrightarrow \ln(x) + 1 \geqslant 0$, *then* $D_{f_8} = \left[\frac{1}{e}, +\infty\right[$

Exercise 0.2 *limits*

$\cdot \lim\limits_{x\to 0} \dfrac{x}{\sqrt{1-x^2}-\sqrt{1+x}} = \frac{0}{0}$, *that is indeterminate forms (IF), we multiply with the conjugate we get*

$$\lim_{x\to 0} \frac{x}{\sqrt{1-x^2}-\sqrt{1+x}} = \lim_{x\to 0} \frac{x\sqrt{1-x^2}+\sqrt{1+x}}{-x^2-x} = \lim_{x\to 0} \frac{\sqrt{1-x^2}+\sqrt{1+x}}{-x-1} = -2$$

$\cdot \lim\limits_{x\to 1} \dfrac{\ln x}{x-1} = \frac{0}{0}$, *(IF), by Hospital rule we get*

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1$$

$\cdot \lim\limits_{x\to 0} \dfrac{\ln\left(1+x^2\right)}{\sin^2 x} = \frac{0}{0}$, *(IF), by Hospital rule we get*

$$\lim_{x\to 0} \frac{\ln\left(1+x^2\right)}{\sin^2 x} = \lim_{x\to 0} \frac{\frac{2x}{1+x^2}}{2\cos x \sin x} = \lim_{x\to 0} \frac{x}{\sin x} \frac{1}{(1+x^2)\cos x}$$

we have $\lim_{x\to 0}\frac{x}{\sin x}=1$, *sinse* $\lim_{x\to 0}\frac{\sin x}{x}=1$
and $\lim_{x\to 0}\frac{1}{(1+x^2)\cos x}=1$
Hence,

$$\lim_{x\to 0}\frac{\ln\left(1+x^2\right)}{\sin^2 x}=\lim_{x\to 0}\frac{\frac{2x}{1+x^2}}{2\cos x\sin x}=\lim_{x\to 0}\frac{x}{\sin x}\frac{1}{(1+x^2)\cos x}=1$$

$\cdot\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}=\frac{0}{0}$, *(IF), by Hospital rule we get*

$$\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}=\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{2x}=\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{2x}=-\frac{1}{2}$$

$\cdot\lim_{x\to+\infty}\frac{\ln\left(1+e^{2x}\right)}{x}=\frac{\infty}{\infty}$, *(IF), by Hospital rule twice we get*

$$\lim_{x\to+\infty}\frac{\ln\left(1+e^{2x}\right)}{x}=\lim_{x\to+\infty}\frac{\frac{2e^{2x}}{1+e^{2x}}}{1}=\lim_{x\to+\infty}\frac{4e^{2x}}{1+e^{2x}}=2$$

$\cdot\lim_{x\to 4}\frac{3-\sqrt{x+5}}{1-\sqrt{5-x}}=\frac{0}{0}$, *(IF), by Hospital rule we get*

$$\lim_{x\to 4}\frac{3-\sqrt{x+5}}{1-\sqrt{5-x}}=\lim_{x\to 4}\frac{\frac{1}{2\sqrt{x+5}}}{\frac{1}{2\sqrt{5-x}}}=\lim_{x\to 4}\frac{\sqrt{5-x}}{\sqrt{x+5}}=\frac{1}{3}$$

$\lim_{x\to+\infty}\sqrt{x^2+4x+3}-(x+2)=+\infty-\infty$, *that is indeterminate forms (IF), we multiply with the conjugate we get*

$$\lim_{x\to+\infty}\sqrt{x^2+4x+3}-(x+2)=\lim_{x\to+\infty}\frac{-1}{\sqrt{x^2+4x+3}+(x+2)}=0.$$

Exercise 0.3 *I)*

$$f_1(x)=\frac{x^2}{x-2},\ f_2(x)=\ln\left(\frac{2+x}{2-x}\right)$$

f_1 *is a rational function and its domain of definition* $D_{f_1}=\mathbb{R}-\{2\}$, *then* f_1 *is continuous on its domain* $\mathbb{R}-\{2\}$

$f_2(x)$ *is logharithmic function its domain of definition* $D_{f_2}=]-2,+2[$, *then* f_2 *is continuous on its domain* $]-2,+2[$

II)

$$f_1(x)=\frac{1-\cos x}{x^2},\quad f_2(x)=\frac{e^x-e^{-x}}{x}$$

we heve $D_{f_1}=\mathbb{R}-\{0\}$, *then* f_1 *is continuous on* $\mathbb{R}-\{0\}$, *and we have*
$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\frac{1}{2}$

So we can extend by continuity at the point $x_0 = 0$ *the function* f_1 *and we write*

$$h_1(x) = \begin{cases} \frac{1-\cos x}{x^2}, \text{ if } x \neq 0 \\ \frac{1}{2}, \text{ if } x = 0 \end{cases}$$

is the extension by continuity of f_1
Similarly we have f_2 *is continious on its domain* $D_{f_2} = \mathbb{R} - \{0\}$
and $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{x} = 2$, *So we can extend by continuity at the point* $x_0 = 0$ *the function* f_2 *and we write*

$$h_2(x) = \begin{cases} \frac{e^x - e^{-x}}{x}, \text{ if } x \neq 0 \\ 2, \text{ if } x = 0 \end{cases}$$

is the extension by continuity of f_2

Exercise 0.4 *Let f a function defined by*

$$\begin{cases} \dfrac{2x}{1+x^2} \text{ if } x \in [-1, 0[\\ \sqrt{x} \text{ if } x \in [0, 3] \end{cases}$$

1) Determine if the function f is continuous and differentiable at the points: $x_0 = -1$; $x_0 = 0$ *and* $x_0 = 3$:
We have
For continuity
$\lim\limits_{x\to -1} \dfrac{2x}{1+x^2} = -1 = f(-1)$, *then f is continuous at* $x_0 = -1$

$\lim\limits_{x \overset{\prec}{\to} 0} f(x) = \lim\limits_{x \overset{\prec}{\to} 0} \dfrac{2x}{1+x^2} = 0 = f(0)$, *and* $\lim\limits_{x \overset{\succ}{\to} 0} f(x) = \lim\limits_{x \overset{\succ}{\to} 0} \sqrt{x} = 0 = f(0)$
Hence f is continuous at $x_0 = 0$

$\lim\limits_{x\to 3} \sqrt{x} = \sqrt{3} = f(3)$, *then f is continuous at* $x_0 = 3$

for differentiability
$\lim\limits_{x\to -1} \dfrac{\frac{2x}{1+x^2} + 1}{x+1} = \lim\limits_{x\to -1} \dfrac{1+x}{1+x^2} = 0$, *hence f is differentiable in right at a point* $x_0 = -1$

$\lim\limits_{x \overset{\prec}{\to} 0} \dfrac{\frac{2x}{1+x^2}}{x} = 2$ *and* $\lim\limits_{x \overset{\succ}{\to} 0} \frac{\sqrt{x}}{x} = +\infty$, *so hence f is not differentiable at* $x_0 = 0$

$\lim\limits_{x \prec 3} \frac{\sqrt{x}-\sqrt{3}}{x-3} = \frac{1}{2\sqrt{3}}$, *hence f is differentiable in left at a point* $x_0 = 3$

2) From Q1 we deduce that f is continuous on its domain $[-1, 3]$, *and fifferentiable on* $[-1, 0[\cup]0, 3]$

3)

$$f^{'}(x)=\begin{cases}\frac{2-2x^2}{(1+x^2)^2}, \ if \ x\in[-1,0[\\ \dfrac{1}{2\sqrt{x}}, \ if \ x\in]0,3]\end{cases}$$