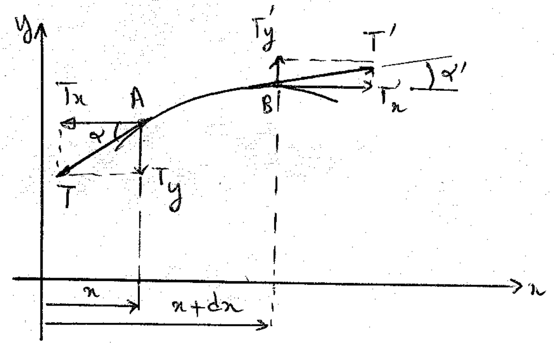
Chapter 2) Vibrating strings

1) Definition: A vibrating string is a one-dimensional, tense medium without stiffness whose movements are studied in the absence of friction.

2) Equation of motion:



Let T be the tension of the string, under equilibrium conditions the string is rectilinear. Assume that we slowly move the string a small amount. We consider a segment AB of the string of length dx which has been displaced by a distance ξ from its equilibrium position.

A tangential force T acts at each end. Due to the curvature of the string, the two forces applied at A and B are not directly opposed.

- Vertical component of each forces :

$$T'_y = T\sin\alpha' \qquad T_y = T\sin\alpha$$

- Resultant forces acting on segment AB:

$$F_y = T(\sin\alpha' - \sin\alpha)$$

As α and α′ are small their sines can be compared to tangents

$$F_y = T(\tan\alpha' - \tan\alpha) = T\,d(\tan\alpha) = T\frac{\partial}{\partial x}(\tan\alpha)\,dx$$

We use the partial derivative because *Tan (α)* depends on both position and time, *Tan (α)* is the slope of the curve that the string follows, therefore:

$$\tan\alpha = \frac{\partial \xi}{\partial x}$$

This force must be equal to : $F_y = T\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right)dx$

equato: $\rho\,dx\,\frac{\partial^2 \xi}{\partial t^2} = T\frac{\partial^2 \xi}{\partial x^2}\,dx$

ρ: linear density of the string (mass per unit length)

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 \xi}{\partial x^2} \qquad (1)$$

The solution to this equation is the superposition of two waves propagating in opposite directions without deformation with the velocity. $V = \sqrt{\frac{T}{\rho}}$

Example: for a rope of linear density ρ=0.04g/cm stretched by a weight of 40 kg we find a speed $V = 320\ ms^{-1}$; speed which is of the same order of magnitude as that of acoustic waves in the air.

3) Additional conditions:

If we fix the position, the initial speed of each point of the rope and the movement of its two ends the movement of any point of the rope will be determined.

There are 2 kinds of conditions:

Initial conditions; for $t=0$

Φ and ψ being two functions defined for $0 \leq x \leq L$

L: length of the rope

$$\begin{cases} \xi(x,0) = \varphi(x) \\ \dfrac{\partial \xi(x,0)}{\partial t} = \psi(x) \end{cases}$$

Boundary conditions: we assume that the string is fixed at both ends

For $\begin{cases} x = 0 \\ x = L \end{cases}$ $\quad \begin{matrix} \xi(0,t) = 0 \\ \xi(L,t) = 0 \end{matrix} \quad \forall\, t \geq 0$

4) Elementary solutions; superposition of solutions:

We are looking for a solution to the equation of the vibrating strings of the form

(method of separation of variables):

$$\xi = f(x) \cdot g(t) \qquad (2)$$

The common value of the 2 members is a negative constant ($-\omega^2$)

$$V^2 g(t)\cdot f''(x) = f(x)\cdot g''(t) \Rightarrow \frac{f''}{f} = \frac{1}{V^2}\,\frac{g''}{g} = -\omega^2$$

The solutions will be: $f''+\omega^2 f=0$ et $g''+V^2\omega^2 g=0$

$$\begin{cases} f = A_0\cos(\omega x) + B_0\sin(\omega x) \\ g = A\cos(V\omega t) + B\sin(V\omega t) \end{cases} \quad (3)$$

Note: If we had chosen a positive constant, the function would have been represented by hyperbolic functions, it would not have been possible to find a function which is bounded (small movement).

Writing boundary conditions:

$\xi=0$ for $x=0$ and $x=L$; So :

$$f(x=0)=f(x=L)=0 \Rightarrow A_0\cos\omega 0 + B_0\sin\omega 0 = 0$$

$$\Rightarrow A_0=0\ ;\ f=B_0\sin\omega L=0 \Rightarrow \omega L = n\pi$$

$$\omega=\frac{n\pi}{L} \quad (4)$$

If we choose $B_0=1$; we can write the elementary solution ξ_n in the form:

$$\xi(x,t) = \left[\sin\frac{n\pi}{L}x\right]\left[A_n\cos\frac{Vn\pi}{L}t + B_n\sin\frac{Vn\pi}{L}t\right] \quad (5)$$

A_n and B_n are constants; correspond to the chosen value ω_n

Any linear superposition of elementary solutions ξ_n is a solution of the vibrating string equation.

$$\xi(x,t) = \sum_{n=1}^{\infty}\left(\sin\frac{n\pi}{L}x\right)\left(A_n\cos\frac{nV\pi}{L}t + B_n\sin\frac{Vn\pi}{L}t\right) \quad (6)$$

It is a Fourier series with respect to x (period $2L$) and with respect to t (period $2L/V$)

Writing initial conditions:

For $t=0$; $\xi = \varphi(x)$ et $\dfrac{\partial \xi}{\partial t} = \psi(x)$

So
$$\begin{cases} \varphi(x) = \sum_{n=1}^{\infty} A_n \sin \dfrac{n\pi}{L} x \\ \dfrac{L}{V\pi} \psi(x) = \sum_{n=1}^{\infty} n B_n \sin \dfrac{n\pi}{L} x \end{cases} \qquad (7)$$

• The A_n are the coefficients of the Fourier series expansion of the odd function with period $2L$ equal to $\phi(x)$ if $0 \leqslant x \leqslant L$

• The B_n are the coefficients of the Fourier series expansion of the odd function with period $2L$ equal to $\psi(x)/(\pi V)$ if $0 \leqslant x \leqslant L$

$$\begin{cases} A_n = \dfrac{2}{L} \displaystyle\int_0^L \varphi(x) \sin \dfrac{n\pi}{L} x \, dx \\ B_n = \dfrac{2}{n V \pi} \displaystyle\int_0^L \psi(x) \sin \dfrac{n\pi}{L} x \, dx \end{cases}$$

Conclusion: for each point x; $\xi(x,t)$ is a periodic function of time with period $2L/V$. The vibration of the string is a superposition of sinusoidal harmonics whose frequencies are a multiple of a fundamental frequency equal to $V/2L= 1/2L\ \ T/\rho = \gamma 1$