
Exercice 01 :

$$\Re = \frac{1}{l}\begin{cases} l = \lim\limits_{n\to+\infty}\left|\dfrac{a_{n+1}}{a_n}\right| \\ l = \lim\limits_{n\to+\infty}\sqrt[n]{|a_n|}\end{cases}$$

1) $\sum\limits_{n\geq 0}\left(\dfrac{x^n}{n!}\right)$ $\quad a_n = \dfrac{1}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \xrightarrow{n\to+\infty} 0 \Rightarrow \Re = +\infty \text{ et } D = \mathbb{R}.$$

2) $\sum\limits_{n\geq 0}\left(\dfrac{n}{2n+1}\right)^{2n-1} x^n$ $\quad a_n = \left(\dfrac{n}{2n+1}\right)^{2n-1}$ $\quad \sqrt[n]{|a_n|} = \left(\dfrac{n}{2n+1}\right)^{\frac{2n-1}{n}}$

$$\sqrt[n]{|a_n|} = \left(\frac{n}{2n+1}\right)^{\frac{2n-1}{n}} \xrightarrow{n\to+\infty} \frac{1}{4} \Rightarrow \Re = 4$$

$i)$ $x = 4$ We have the numerical serie : $\sum\limits_{n\geq 0}\left(\dfrac{n}{2n+1}\right)^{2n-1} 4^n$

$$u_n = \left(\frac{n}{2n+1}\right)^{2n-1} 4^n = \frac{2n+1}{n}\left(\frac{2n}{2n+1}\right)^{2n} = \frac{2n+1}{n}\frac{1}{\left(1+\frac{1}{2n}\right)^{2n}} \xrightarrow{n\to+\infty} \frac{2}{e} \neq 0$$

Then : $\sum\limits_{n\geq 0}\left(\dfrac{n}{2n+1}\right)^{2n-1} 4^n$ Diverges.

$ii)$ $x = -4$ We have : $\sum\limits_{n\geq 0}\left(\dfrac{n}{2n+1}\right)^{2n-1} (-4)^n$

$$\lim_{n\to+\infty}\left(\frac{n}{2n+1}\right)^{2n-1}(-4)^n = \lim_{n\to+\infty}(-1)^n 4^n\left(\frac{n}{2n+1}\right)^{2n-1} \quad \text{Don't exist.}$$

$$\begin{pmatrix} 4^n \left(\frac{n}{2n+1}\right)^{2n-1} \xrightarrow{n\to+\infty} 2e^{-1} \\ (-1)^n \xrightarrow{n\to+\infty} \pm 1 \end{pmatrix} \quad \text{Then: } \sum_{n\geq 0} \left(\frac{n}{2n+1}\right)^{2n-1} (-4)^n \text{ Diverges.}$$

Then : $D =]-4 \quad 4[$.

2) $\sum_{n\geq 0} \left(\frac{n}{n+1}\right)\left(\frac{x}{2}\right)^n \qquad a_n = \left(\frac{n}{n+1}\right)2^{-n} \qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{n(n+2)}2^{-1} \xrightarrow{n\to+\infty} 2^{-1} \Rightarrow \mathfrak{R} = 2.$

i) $x = 2 \qquad \sum_{n\geq 0} \frac{n}{n+1}$

$$u_n = \frac{n}{n+1} \xrightarrow{n\to+\infty} 1 \neq 0\,, \text{ Then : } \sum_{n\geq 0} \frac{n}{n+1} \quad \text{is divergent.}$$

ii) $x = -2 \qquad \sum_{n\geq 0} \frac{n}{n+1} \,(-1)^n$

$$\lim_{n\to+\infty} \frac{n}{n+1} \,(-1)^n = \pm 1 \neq 0\,, \text{ then: } \sum_{n\geq 0} \frac{n}{n+1}(-1)^n \quad \text{is divergent.}$$

Then : $D =]-2 \quad 2[$.

3) $\sum_{n\geq 0} \left(\frac{3n+5}{2+n}\right)^n x^n \qquad a_n = \left(\frac{3n+5}{2+n}\right)^n, \quad \sqrt[n]{|a_n|} = \frac{3n+5}{2+n} \xrightarrow{n\to+\infty} 3 \Rightarrow \mathfrak{R} = \frac{1}{3}.$

i) $x = \frac{1}{3} \qquad \sum_{n\geq 0} \left(\frac{3n+5}{n+2}\right)^n \frac{1}{3^n} = \sum_{n\geq 0} \left(\frac{3n+5}{3n+6}\right)^n$

$$u_n = \left(\frac{3n+5}{3n+6}\right)^n = \left(\frac{1+\frac{5}{3n}}{1+\frac{6}{3n}}\right)^n = \frac{\left(1+\frac{5}{3n}\right)^n}{\left(1+\frac{6}{3n}\right)^n} \xrightarrow{n\to+\infty} e^{-\frac{1}{3}} \neq 0, \text{ then : } \sum_{n\geq 0} \left(\frac{3n+5}{n+2}\right)^n \frac{1}{3^n} \text{ is divergent.}$$

ii) $x = -\frac{1}{3} \qquad \sum_{n\geq 0} \left(\frac{3n+5}{n+2}\right)^n \frac{(-1)^n}{3^n}$

$$\lim_{n\to+\infty} \left(\frac{3n+5}{n+2}\right)^n \frac{(-1)^n}{3^n} \text{ Don't exist , Then : } \sum_{n\geq 0} \left(\frac{3n+5}{n+2}\right)^n \frac{(-1)^n}{3^n} \quad \text{diverges.}$$

$$Then\ :\ D = \left]-\frac{1}{3} \quad \frac{1}{3}\right[.$$

4) $\sum_{n\geq 0}\frac{1}{3^n}x^n \qquad a_n=\frac{1}{3^n} \qquad \sqrt[n]{|a_n|}=\frac{1}{3}$ Then : $\Re = 3$.

$i)\ x=3 \quad \sum_{n\geq 0} 1$ Diverges.

$ii)\ x=-3 \quad \sum_{n\geq 0} (-1)^n$ Diverges.

Then : $D =]-3\ \ 3[$.

5) $\sum_{n\geq 0}(1+n)x^n \qquad a_n=(1+n)\ , \qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{n+2}{n+1} \xrightarrow{n\to+\infty} 1$ Then : $\Re = 1$.

$i)\ x=1 \quad \sum_{n\geq 0}(1+n)$ Diverges. $\left((1+n)\xrightarrow{n\to+\infty} +\infty \neq 0\right)$

$ii)\ x=-1 \quad , \sum_{n\geq 0}(1+n)(-1)^n$ Diverges. $\left(\lim_{n\to+\infty}(1+n)\ (-1)^n = \pm\infty\right)$

Then : $D =]-1\ \ 1[$.

Exercice 02 :

1) $\sum_{n\geq 0} n^2x^n \qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{n^2} \xrightarrow{n\to+\infty} 1$ Then : $\Re = 1$.

We have : $\frac{1}{1-x}=\sum_{n\geq 0}x^n \Rightarrow \left(\frac{1}{1-x}\right)' = \left(\sum_{n\geq 0}x^n\right)' = \sum_{n\geq 1}nx^{n-1} \Rightarrow \frac{1}{(1-x)^2}=\sum_{n\geq 1}nx^{n-1}$

$$\Rightarrow \quad \frac{1}{(1-x)^2}=\sum_{n\geq 1}nx^{n-1}=\sum_{n\geq 0}(n+1)x^n=\sum_{n\geq 0}nx^n+\sum_{n\geq 1}x^n=\sum_{n\geq 0}nx^n+\frac{1}{1-x}$$

$$\frac{1}{(1-x)^2}-\frac{1}{1-x}=\frac{x}{(1-x)^2}=\sum_{n\geq 0}nx^n \text{ (with derivation and multiplication by } x) \Rightarrow \frac{(1+x)x}{(1-x)^3}=\sum_{n\geq 1}n^2x^n$$

$$\sum_{n\geq 0}n^2x^n=\frac{(1+x)x}{(1-x)^3}.$$

2) $\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n \qquad \Re = 1.$

We have : $\sum_{n\geq 0} (-1)^n x^n = \frac{1}{1+x}$ By integration term by term : $\sum_{n\geq 0} \frac{(-1)^n}{n+1} x^{n+1} = ln(|1+x|)$

$$\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n = ln(|1+x|).$$

3) $\sum_{n\geq 1} \frac{x^n}{n(n+1)} \qquad \Re = 1.$

We have : $\sum_{n\geq 0} x^n = \frac{1}{1-x}$ By integration $\sum_{n\geq 0} \frac{1}{n+1} x^{n+1} = -ln(|1-x|)$.

$$\sum_{n\geq 1} \frac{x^n}{n} = -ln(|1-x|).$$

Exercice 03 :

1) $f(z) = \frac{x^2}{1-x} = \sum_{n\geq 2} x^n = \sum_{n\geq 0} x^{n+2}$

2) $h(z) = \frac{1}{(1+x)^2}$ We have : $\frac{1}{1+x} = \sum_{n\geq 0} (-1)^n x^n$ By derivation : $\frac{-1}{(1+x)^2} = \sum_{n\geq 1} (-1)^n n x^{n-1}$

$$h(z) = \frac{1}{(1+x)^2} = \sum_{n\geq 1} (-1)^{n+1} n x^{n-1} = \sum_{n\geq 0} (-1)^n (n+1) x^n .$$

3) $g(z) = \frac{2}{x(x^2-4)} = \frac{1}{2x}\left(\frac{1}{x-2} - \frac{1}{x+2}\right) = \frac{1}{2x}\left(-\frac{1}{2}\sum_{n\geq 0} \frac{x^{\,n}}{2^n} + \frac{1}{2}\sum_{n\geq 0} (-1)^n \frac{x^n}{2^n}\right)$

$$g(z) = \sum_{n\geq 0} ((-1)^n - 1) \frac{x^{n-1}}{2^{n+2}} = \sum_{n\geq 0} ((-1)^n - 1) \frac{x^{n-1}}{2^{n+2}}.$$