
Exercise 01 :

a) By the characteristics : $(E_1): \begin{cases} y\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y} - 2u = 0 \\ u(x,0) = x \end{cases}$ (Rectifier la condition initiale)

The characteristic system : $\dfrac{dx}{A} = \dfrac{dy}{B} = \dfrac{du}{C} \Rightarrow \dfrac{dx}{y} = \dfrac{dy}{-1} = \dfrac{du}{2u}$

$$\begin{cases} \dfrac{dx}{y} = \dfrac{dy}{-1} \\ \dfrac{dy}{-1} = \dfrac{du}{2u} \end{cases} \Rightarrow \begin{cases} ydy = -dx \\ -2y + C = ln|u| \end{cases} \Rightarrow \begin{cases} \dfrac{1}{2}y^2 = -x + C_1 \\ u = e^{-2y+C} \end{cases} \Rightarrow \begin{cases} C_1 = x + \dfrac{1}{2}y^2 \\ C_2 = e^C = u\, e^{2y} \end{cases}$$

Then : $\varphi(C_1, C_2) = \varphi\left(x + \dfrac{1}{2}y^2, u\, e^{2y}\right) = 0$ or $u(x,y) = e^{-2y} f\left(x + \dfrac{1}{2}y^2\right)$, f arbitrary function.

$$u(x,0) = x \Rightarrow f(x) = x \Rightarrow u(x,y) = e^{-2y}\left(x + \dfrac{1}{2}y^2\right).$$

$(E_2) : \begin{cases} \dfrac{\partial u}{\partial x} - x^2\dfrac{\partial u}{\partial y} = 0 \\ u(x,0) = x^3 \end{cases}$ characteristic system : $\dfrac{dx}{1} = \dfrac{dy}{-x^2} = \dfrac{du}{0}$

$$\begin{cases} \dfrac{dx}{1} = \dfrac{dy}{-x^2} \\ u = C_2 \end{cases} \Rightarrow \begin{cases} -x^2dx = dy \\ u = C_2 \end{cases} \Rightarrow \begin{cases} C_1 = y + \dfrac{1}{3}x^3 \\ C_2 = u \end{cases} \Rightarrow u(x,y) = f\left(y + \dfrac{1}{3}x^3\right) ,\ f \text{ arbitrary.}$$

$$u(x,0) = x^3 \Rightarrow f\left(\dfrac{1}{3}x^3\right) = x^3 \Rightarrow f(x) = 3x \Rightarrow u(x,y) = 3y + x^3 .$$

b) By separation of variables : $(E_1): \begin{cases} 2x\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0 \\ u(1,0) = 1 \end{cases}$ Put : $u(x,y) = X(x)Y(y)$

$$u = XY \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = X'Y \\ \dfrac{\partial u}{\partial y} = XY' \end{cases} \Rightarrow (E_1):\ 2xX'Y + XY' = 0 \Rightarrow \dfrac{Y'}{Y} = -2x\dfrac{X'}{X} = k \text{ (Constante)}$$

$$\begin{cases} \dfrac{Y'}{Y} = k \\ -2x\dfrac{X'}{X} = k \end{cases} \Rightarrow \begin{cases} ln|Y| = ky + C_1 \\ ln|X| = -\dfrac{k}{2}\ln(x) + C_2 \end{cases} \Rightarrow \begin{cases} Y = C_3 e^{ky} \\ X = \dfrac{C_4}{\sqrt{x}^k} \end{cases}, \qquad (e^{C_1} = C_3, e^{C_2} = C_4)$$

$$u(x,y) = X(x)Y(y) = C_3 C_4 \frac{e^{ky}}{\sqrt{x}^k} = C\frac{e^{ky}}{\sqrt{x}^k}.$$

$$u(1,0) = 1 \Rightarrow C = 1 \Rightarrow u(x,y) = \frac{e^{ky}}{\sqrt{x}^k}.$$

$$(\mathrm{E}_2): \begin{cases} \dfrac{\partial u}{\partial x} + sin(x)\dfrac{\partial u}{\partial y} = 0 \\ u\left(\dfrac{\pi}{2}, 0\right) = -2 \end{cases} \qquad \text{Put: } u(x,y) = X(x)Y(y)$$

$$u = XY \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = X'Y \\ \dfrac{\partial u}{\partial y} = XY' \end{cases} \Rightarrow (E_2):\ X'Y + sin(x)XY' = 0 \Rightarrow \frac{Y'}{Y} = -\frac{X'}{Xsin(x)} = k$$

$$\begin{cases} \dfrac{Y'}{Y} = k \\ -\dfrac{X'}{Xsin(x)} = k \end{cases} \Rightarrow \begin{cases} ln|Y| = ky + C_1 \\ ln|X| = kcos(x) + C_2 \end{cases} \Rightarrow \begin{cases} Y = C_1 e^{ky} \\ X = C_2 e^{kcos(x)} \end{cases} \qquad (e^{C_1} \to C_1, e^{C_2} \to C_2)$$

The general solution is : $u(x,y) = X(x)Y(y) = Ce^{k(y+cos(x))}$. ($C_1C_2 = C$)

$$u\left(\frac{\pi}{2}, 0\right) = -2 \Rightarrow C = -2 \Rightarrow u(x,y) = -2e^{k(y+cos(x))}$$. (particular solution)

c) By the coordinates method : $(\mathrm{E}_1): \begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 1 \\ u(2x, x) = x \end{cases}$ Put : $\begin{cases} s = x + y \\ t = x - y \end{cases}$

$$\left(A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = C \ \text{ On Pose } \begin{cases} s = Ax + By \\ t = Bx - Ay \end{cases}\right)$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial x} = \dfrac{\partial u}{\partial s} + \dfrac{\partial u}{\partial t} \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial y} + \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial y} = \dfrac{\partial u}{\partial s} - \dfrac{\partial u}{\partial t} \end{cases} \qquad (\mathrm{E}_1): 2\frac{\partial u}{\partial s} = 1 \Rightarrow u(x,y) = \frac{1}{2}s + f(t), \qquad f\text{Arbitrary.}$$

The general solution is : $u(x,y) = \dfrac{1}{2}(x+y) + f(x-y)$.

$u(2x, x) = x \Rightarrow f(x) = -\dfrac{1}{2}x \Rightarrow u(x,y) = y$.

$$(E_2):\begin{cases}\dfrac{\partial u}{\partial x}-3\dfrac{\partial u}{\partial y}=0\\ u(0,y)=\sin(y)\end{cases},\quad \text{Posons}:\begin{cases}s=x-3y\\ t=-3x-y\end{cases}\quad \text{D'où}\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x}+\dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial x}=\dfrac{\partial u}{\partial s}-3\dfrac{\partial u}{\partial t}\\ \dfrac{\partial u}{\partial y}=\dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial y}+\dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial y}=-3\dfrac{\partial u}{\partial s}-\dfrac{\partial u}{\partial t}\end{cases}$$

$(E_2): 10\dfrac{\partial u}{\partial s}=0 \Rightarrow \dfrac{\partial u}{\partial s}=0 \Rightarrow u=f(t),$ f arbitrary Function.

Then : $u(x,y)=f(-3x-y)$.

$$u(0,y)=sin(y)\Rightarrow f(-y)=sin(y)\Rightarrow f(x)=-sin(x)$$
$$u(x,y)=sin(3x+y).$$

Exercise 02 :

$$(E_1):\begin{cases}\dfrac{\partial^2 u}{\partial x\partial y}=xe^y\\ u(x,0)=3x^2+5\\ u(0,y)=-y^3+5\end{cases}$$

$$\frac{\partial^2 u}{\partial x\partial y}=xe^y \Rightarrow \frac{\partial u}{\partial x}=\int xe^y\,dy=xe^y+f(x) \Rightarrow u(x,y)=\frac{1}{2}x^2e^y+F(x)+g(y)$$

where : f and g two arbitrary functions.

$$\begin{cases}u(x,0)=3x^2+5\\ u(0,y)=-y^3+5\end{cases} \Rightarrow \begin{cases}\frac{1}{2}x^2+F(x)+g(0)=3x^2+5\\ F(0)+g(y)=-y^3+5\end{cases} \Rightarrow \begin{cases}F(x)=\frac{5}{2}x^2+5-g(0)\\ g(y)=-y^3+g(0)\end{cases}$$

$$F(x)+g(y)=\frac{5}{2}x^2-y^3+5 \Rightarrow u(x,y)=\frac{1}{2}x^2e^y+\frac{5}{2}x^2-y^3+5$$

$(E_2):\ \dfrac{\partial^2 u}{\partial x^2}+\dfrac{1}{x}\dfrac{\partial u}{\partial x}=0$ Put : $\dfrac{\partial u}{\partial x}=v,\ \dfrac{\partial v}{\partial x}=\dfrac{\partial^2 u}{\partial x^2}$

$$(E_2):\ \frac{\partial v}{\partial x}+\frac{1}{x}v=0 \Rightarrow \frac{1}{v}\frac{\partial v}{\partial x}=-\frac{1}{x} \Rightarrow v=\frac{1}{x}f(y)$$

$\dfrac{\partial u}{\partial x}=v=\dfrac{1}{x}f(y)\Rightarrow u(x,y)=f(y)ln|x|+g(y).$ f , g arbitrary functions.

Exercise 03 :

$$(E_1):\ \frac{\partial^2 u}{\partial x^2} - 4\,\frac{\partial^2 u}{\partial y^2} = y^2$$

1) La forme canonique :

($\Delta = B^2 - 4AC = 4 > 0$ Alors (E_1) is hyperbolic equation.)

Solve the characteristic equation : $A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$

$$\left(\frac{dy}{dx}\right)^2 - 4 = 0 \Rightarrow \frac{dy}{dx} = \pm 2 \quad , \quad \begin{cases} \frac{dy}{dx} = -2 \\ \frac{dy}{dx} = 2 \end{cases} \Rightarrow \begin{cases} C_1 = y + 2x \\ C_2 = y - 2x \end{cases}$$

$$\text{Put : } \begin{cases} s = y + 2x \\ t = y - 2x \end{cases} , \quad \begin{cases} \frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial s} - 2\frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 u}{\partial x^2} = 4\frac{\partial^2 u}{\partial s^2} - 8\frac{\partial^2 u}{\partial s \partial t} + 4\frac{\partial^2 u}{\partial t^2} \\ \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial s^2} + 2\frac{\partial^2 u}{\partial s \partial t} + \frac{\partial^2 u}{\partial t^2} \end{cases}$$

$\frac{\partial^2 u}{\partial s \partial t} = -\frac{1}{64}(s+t)^2$ is the canonical form of (E_1).

$$\frac{\partial^2 u}{\partial s \partial t} = -\frac{1}{64}(s+t)^2 \Rightarrow \frac{\partial u}{\partial t} = -\frac{1}{64}\frac{1}{3}(s+t)^3 + f(t) \Rightarrow$$

$$u = -\frac{1}{64}\frac{1}{3}\frac{1}{4}(s+t)^4 + F(t) + g(s) \Rightarrow u(x,y) = -\frac{1}{48}y^4 + F(y-2x) + g(y+2x).$$

$$(E_2):\ \frac{\partial^2 u}{\partial x^2} - 6\,\frac{\partial^2 u}{\partial x \partial y} + 9\,\frac{\partial^2 u}{\partial y^2} = xy^2$$

($\Delta = 0$ Alors (E_2) est parabolique.)

Characteristic equation : $\left(\frac{dy}{dx}\right)^2 + 6\frac{dy}{dx} + 9 = 0 \Rightarrow \left(\frac{dy}{dx} + 3\right)^2 = 0$

$\frac{dy}{dx} + 3 = 0 \Rightarrow C_1 = y + 3x$

Choose C_2 so that the jacobien determinant : $J(C_1, C_2) \neq 0$

Take : $C_2 = x \quad \left(J = \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0\right)$

Put the change of variable : $\begin{cases} s = y + 3x \\ t = x \end{cases}$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial x} = 3\dfrac{\partial u}{\partial s} + \dfrac{\partial u}{\partial t} \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial y} + \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial y} = \dfrac{\partial u}{\partial s} \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 u}{\partial x^2} = 9\dfrac{\partial^2 u}{\partial s^2} + 6\dfrac{\partial^2 u}{\partial s \partial t} + \dfrac{\partial^2 u}{\partial t^2} \\ \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial s^2} \\ \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial s}\right) = 3\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial s \partial t} \end{cases}$$

$\dfrac{\partial^2 u}{\partial t^2} = xy^2 \Rightarrow \dfrac{\partial^2 u}{\partial t^2} = t(s-3t)^2$ is the canonical form of (E_2).

$$\frac{\partial^2 u}{\partial t^2} = t(s-3t)^2 \Rightarrow \frac{\partial u}{\partial t} = \int t(s-3t)^2 dt = \frac{s^2t^2}{2} - 2st^3 + \frac{9}{4}t^4 + f(s), \qquad f \text{ Arbitrary.}$$

$$\Rightarrow u = \frac{s^2t^3}{6} - \frac{1}{2}st^4 + \frac{9}{20}t^5 + tf(s) + g(s), \qquad f, g \text{ Arbitrary functions.}$$

Then : the general solution of (E_2) is :

$$u(x,y) = \frac{(y+3x)^2x^3}{6} - \frac{1}{2}(y+3x)x^4 + \frac{9}{20}x^5 + xf(y+3x) + g(y+3x), \qquad f, g \text{ Arbitrary}$$

$u(0,y) = 0 \Rightarrow g(y) = 0$

$$u(x,0) = 0 \Rightarrow \frac{9}{20}x^5 + xf(3x) = 0 \Rightarrow f(x) = -\frac{1}{180}x^4 .$$

$$\text{Then : } u(x,y) = \frac{(y+3x)^2x^3}{6} - \frac{1}{2}(y+3x)x^4 + \frac{9}{20}x^5 - \frac{1}{180}x(y+3x)^4.$$

Exercice 04 :

1) $\sum_{n\geq 1} \dfrac{n^n}{2^n n!} \qquad u_n = \dfrac{n^n}{2^n n!} \qquad$ D'Alembert

we have : $\left|\dfrac{u_{n+1}}{u_n}\right| = \dfrac{(n+1)^{n+1}}{2^{n+1}(n+1)!}\,\dfrac{2^n n!}{n^n} = \dfrac{1}{2}\left(\dfrac{n+1}{n}\right)^n \xrightarrow{n\to+\infty} \dfrac{e}{2} > 1 \Rightarrow \sum_{n\geq 1}\dfrac{n^n}{2^n n!}$ is divergent .

2) $\sum_{n\geq 1} n ln\left(1+\frac{1}{n}\right)$ $\qquad u_n = n ln\left(1+\frac{1}{n}\right) = ln\left(1+\frac{1}{n}\right)^n$

$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} ln\left(1+\frac{1}{n}\right)^n = 1 \neq 0 \quad \Rightarrow \quad \sum_{n\geq 1} n ln\left(1+\frac{1}{n}\right)$ est divergente.

3) $\sum_{n\geq 1}\left(\frac{2}{3}\right)^{n+1} = \frac{2}{3}\sum_{n\geq 1}\left(\frac{2}{3}\right)^{n}$ Geometric serie $q < 1$, converges .

$$\left(\text{Or : } \quad u_n = \left(\frac{2}{3}\right)^{n+1} \qquad \text{Cauchy : } \sqrt[n]{|u_n|} = \left(\frac{2}{3}\right)^{\frac{n+1}{n}} \xrightarrow{n\to+\infty} \frac{2}{3} < 1 \qquad \sum_{n\geq 1}\left(\frac{2}{3}\right)^{n+1} \text{ converges.} \right)$$

4) $\sum_{n\geq 0}\left(\frac{n}{n+1}\right)^{n^2}$ $\qquad u_n = \left(\frac{n}{n+1}\right)^{n^2}$; $\sqrt[n]{|u_n|} = \left(\frac{n}{n+1}\right)^{n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \xrightarrow{n\to+\infty} \frac{1}{e} < 1$

$\sum_{n\geq 1} u_n$ convergente.

5) $\sum_{n\geq 0}\left(\sqrt{n^3-1}-\sqrt{n^3}\right)$ $\qquad u_n = \sqrt{n^3-1}-\sqrt{n^3} = \frac{1}{\sqrt{n^3-1}+\sqrt{n^3}} \sim \frac{1}{\sqrt{n^3}}$

$\sum_{n\geq 1}\frac{1}{\sqrt{n^3}}$ Riemann serie $\alpha=\frac{3}{2}>1$ converges. Then $\sum_{n\geq 1}\left(\sqrt{n^3-1}-\sqrt{n^3}\right)$ is convergent.

6) $\sum_{n\geq 0}\left(\frac{n}{2n+1}\right)^{n}$ $\qquad u_n = \left(\frac{n}{2n+1}\right)^{n}$; $\sqrt[n]{|u_n|} = \frac{n}{2n+1} \xrightarrow{n\to+\infty} \frac{1}{2} < 1$

Then : $\sum_{n\geq 1} u_n$ converges.

7) $\sum_{n\geq 0}\frac{2n^2}{n(n+2)}$ $\qquad \lim_{n\to+\infty}\frac{2n^2}{n(n+2)} = 2 \neq 0$ Then $\sum_{n\geq 1} u_n$ est divergente.

Exercise 05 :

1) $\sum_{n\geq 1} cos\left(\frac{1}{n}\right)$, $u_n = cos\left(\frac{1}{n}\right) \xrightarrow{n\to+\infty} 1 \neq 0$; Then : $\sum_{n\geq 1} cos\left(\frac{1}{n}\right)$ is divergent.

2) $\sum_{n\geq 0} sin\left(\frac{4}{3^n}\right)$, $u_n = sin\left(\frac{4}{3^n}\right)$ Put : $x = \frac{4}{3^n} \xrightarrow{n\to+\infty} 0$, $\sin(x) \sim x$.

$$u_n = sin\left(\frac{4}{3^n}\right) \sim \frac{4}{3^n}$$

$\sum_{n\geq 0} 4\left(\frac{1}{3}\right)^n$ Geometric serie $q = \frac{1}{3} < 1$ converges. Then : $\sum_{n\geq 0} sin\left(\frac{4}{3^n}\right)$ is convergent.

3) $\sum_{n\geq 1}\left(1 - cos\left(\frac{1}{n}\right)\right)$, $u_n = 1 - cos\left(\frac{1}{n}\right)$ Put : $x = \frac{1}{n} \xrightarrow{n\to+\infty} 0$, $cos(x) \sim 1 - \frac{1}{2}x^2$

$u_n = 1 - \left(1 - \frac{1}{2}\frac{1}{n^2}\right) = \frac{1}{2}\frac{1}{n^2}$ and $\sum_{n\geq 1}\frac{1}{2}\frac{1}{n^2}$ Serie of Riemann $\alpha = 2 > 1$ converges.

Then : $\sum_{n\geq 1}\left(1 - cos\left(\frac{1}{n}\right)\right)$ is convergent.

4) $\sum_{n\geq 1}(-1)^n sin\left(\frac{1}{n^2}\right)$, $u_n = (-1)^n sin\left(\frac{1}{n^2}\right)$

$sin\left(\frac{1}{n^2}\right) \sim \frac{1}{n^2}$ $\sum_{n\geq 1}|u_n| \sim \sum_{n\geq 1}\frac{1}{n^2}$ Serie of Riemann converges $\alpha = 2 > 1$.

Then : $\sum_{n\geq 1}|u_n|$ converges $\Rightarrow \sum_{n\geq 1} u_n$ converges.