If a sinusoidal voltage (or current) is imposed on a network, a sinusoidal response of the same frequency as the applied voltage (or current) appears, in addition to the transient regime. When the transient regime has disappeared, this sinusoidal response remains: it is the permanent sinusoidal regime. In this part, we study linear circuits in which the signals imposed by the generators are sinusoidal.

1. Sinusoidal quantities

A signal is said to be sinusoidal if it is of the form : $X(t) = X_m \cos(\omega t + \varphi) = X\sqrt{2}\cos(\omega t + \varphi)$

X_m: Amplitude of the signal.

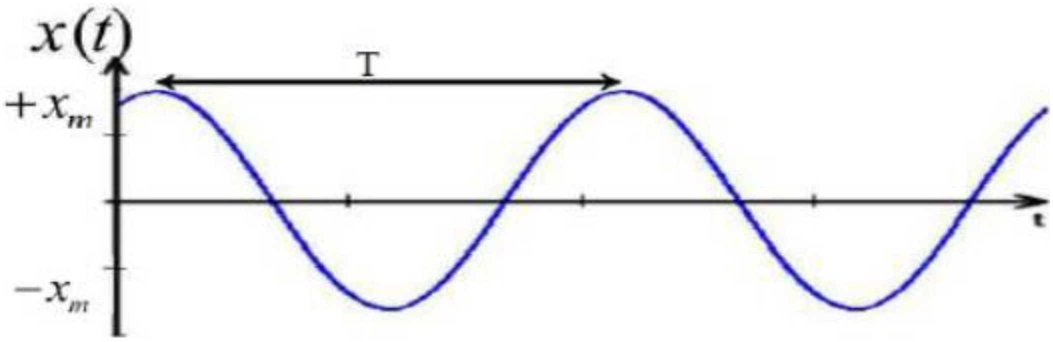
ω : pulsation of the periodic signal and is expressed in (rad/s).

$T = \frac{2\pi}{\omega}$: Signal period, $f = \frac{1}{T} = \frac{\omega}{2\pi}$: Signal frequency.

$\omega t + \varphi$: is the phase of the signal and is expressed in radians (rad).

φ : Initial phase of the signal (at t = 0).

X : Effective value defined by : $X^2 = \frac{1}{T}\int_0^T x^2(t)dt$, we obtain $X = \frac{x_m}{\sqrt{2}}$



2. Representations of sinusoidal quantities

2.1. Vector representation (Fresnel method)

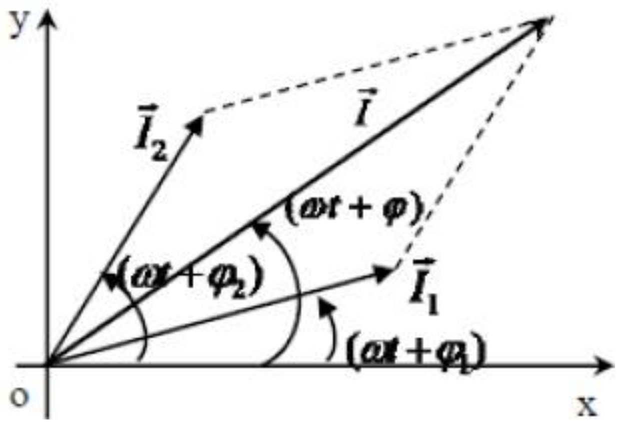
This method allows the addition of instantaneous sinusoidal quantities of the same frequency, but of different amplitudes and phases.

- Consider two sinusoidal currents :

$i_1(t) = I_{m1}\cos(\omega t + \varphi_1)$ and $i_2(t) = I_{m2}\cos(\omega t + \varphi_1)$

The sum of the two currents is : $i(t) = i_1(t) + i_2(t)$.

- To find i (t), we can proceed graphically :



- Consider a vector denoted $\vec{I_1}$ *of* I_{m1} standard, rotating in the plane xOy at an angular velocity ω, and whose angle with the Ox axis at a time t is equal to $\omega t + \varphi$. We define a vector in the same way $\overrightarrow{I_2}$

- The projections on Ox of the vectors $\overrightarrow{I_1}$ and $\overrightarrow{I_2}$ are equal to the currents i_1 and i_2 respectively.

The sum of the two currents i (t) is the projection of the sum vector :

$i(t)$=$I_m$cos(ωt+φ) et $\vec{I} = \overrightarrow{I_1} + \overrightarrow{I_2}$, such as $\|\vec{I}\| = I_m$

2.2. Complex representation

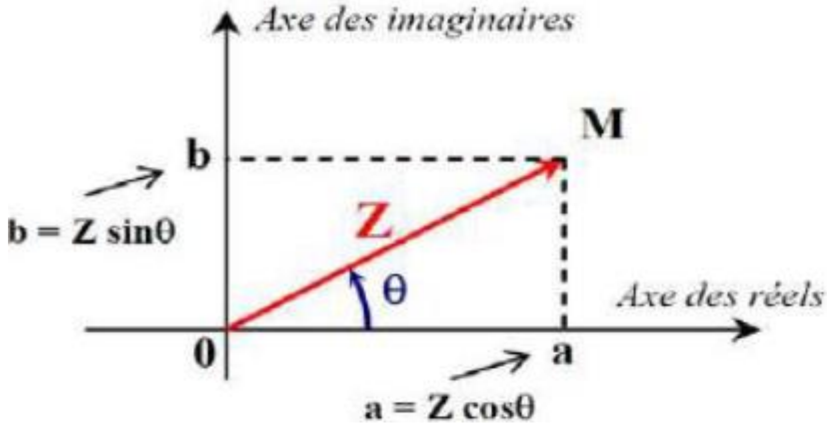
2.2.1. Mathematical reminders

- A complex number can be put in the form: Z = a + jb

We call: a = R(Z) the real part.

and b = Im(Z) the imaginary part.

- We can associate a vector $\overrightarrow{OM}$ in the complex plane: Z = r cosθ + j r sinθ



r = |Z| = $\sqrt{a^2 + b^2}$: modulus of the complex number

θ = arg(Z) = arctan $\frac{b}{a}$: argument (angle) of the complex number

- It can also be written in the exponential form: Z = r$e^{j\theta}$
- or in polar form: Z = [r, θ] = r ∠θ
- special case: J = $e^{j\frac{\pi}{2}}$

2.2.2. Application to sinusoidal signals

A sinusoidal signal x (t) is associated with a complex time quantity X :

$$x(t) = x_m \cos(\omega t + \varphi) = R\left(\underline{X} e^{j\omega t}\right), \qquad \underline{X} = x_m e^{j\varphi} = X\sqrt{2}. e^{j\varphi}$$

$$x(t) \Leftrightarrow \underline{X}$$

X_m : modulus of the complex quantity (|X|),
φ : argument of the complex quantity (argX),
X = $X_m/\sqrt{2}$: effective value.

Note : We will note x(t) as the instantaneous value, X as the RMS value, and $\underline{X}$ as the complex value.

2.2.3. Derivative and integration

Let the function x (t) = xm cos(ω t +φ), the derivative is written :

$$\frac{dx}{dt} = -x_m \omega sin(\omega\, t\, + \varphi\,) = x_m cos\left(\omega\, t\, + \varphi + \frac{\pi}{2}\right)$$

It is associated with the complex amplitude :

$$\omega x_m e^{j\left(\omega\, t\, + \varphi + \frac{\pi}{2}\right)} = j\omega x_m e^{j(\omega\, t\, + \varphi\,)} = j\omega \underline{X} e^{j\omega t}$$

Therefore : $\frac{dx}{dt} = j\omega \underline{X}$

In the same way, it is shown that the integral is equivalent to dividing by jω: $\int x dx = \frac{1}{j\omega}\underline{X}$

3. Complex model of a circuit in the sinusoidal regime

In a sinusoidal circuit, we can write :

$$e(t) = E_0 cos\omega t = R(\underline{E} e^{j\omega t}), \ \underline{E} = E_0$$

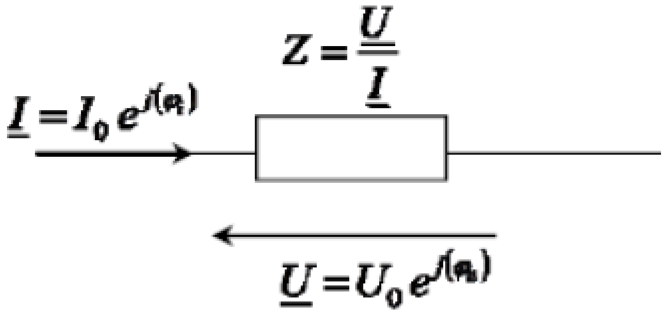
The voltage source e(t) is replaced by its complex form denoted E :

$$e(t) \Leftrightarrow \underline{E} = E_0$$

In the complex model, any linear dipole has a complex impedance : Z = R + jX
where R: represents the resistance of the dipole.
X: reactance.

3.1. Complex impedances of elementary dipoles

The complex impedance Z is defined for a linear dipole as being equal to the ratio of the value complex of the voltage $\underline{U}$ on the complex value of the current $\underline{I}$: $Z = \frac{\underline{U}}{\underline{I}}$



3.1.1. Impedance of a resistor

we have: u(t) = R i (t)

Moving on to the complex amplitudes, we then obtain: $\underline{U}$ = Z.$\underline{I}$

In the case of a resistor, the complex impedance is equal to R : Z_R = R.

3.1.2. Impedance of an ideal coil

The relationship between current and voltage across an inductor coil L is: u(t) = $L\frac{di(t)}{dt}$

This temporal relationship is expressed in terms of complex amplitudes by : $\underline{U} = jL\omega\underline{I}$

The definition of the complex impedance of a linear dipole then allows us to set Z_L = jLω

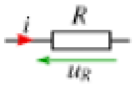
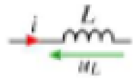
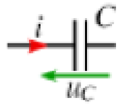
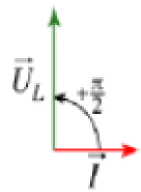
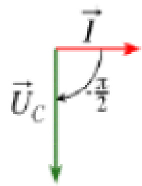
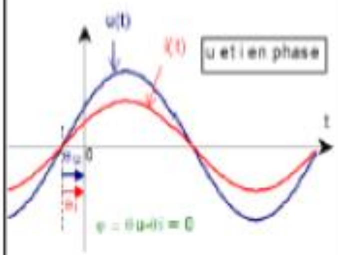
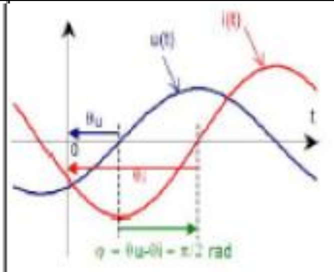
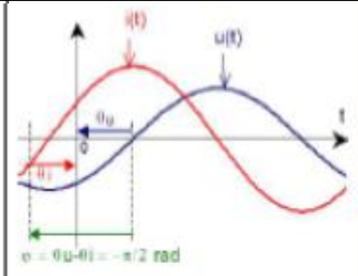
3.1.3. Impedance of a capacitor

The relationship between current and voltage across an ideal capacitor of capacitance C is $U(t) = \frac{1}{C}\int i(t)dt$. We deduct $\underline{U} = \frac{1}{jC\omega}\underline{I}$

The expression of the capacitor impedance is written: $Z_C = \frac{1}{jC\omega}$

3.2. Sinusoidal Laws

All laws seen in a continuous regime are applicable to sinusoidal regimes provided that they are Apply to snapshot values or complex values.

	Résistance R	Inductance L	Capacité C
Schéma			
Equation fondamentale	$u_R(t) = R\,i(t)$	$u_L(t) = L\dfrac{di(t)}{dt}$	$u_C(t) = \dfrac{1}{C}\int i(t)dt$
Equation complexe	$\underline{U_R} = R\,\underline{I}$	$\underline{U_L} = jL\omega\,\underline{I}$	$\underline{U_C} = \dfrac{1}{jC\omega}\underline{I}$
Impédance Z (Ω)	$Z_R = R$	$Z_L = jL\omega$	$Z_C = \dfrac{-j}{C\omega}$
Admittance Y (S)	$Y_R = \dfrac{1}{R}$	$Y_L = -\dfrac{j}{L\omega}$	$Y_C = jC\omega$
Déphasage $\varphi(rad) = \Delta\varphi = \varphi_u - \varphi_i$	$\varphi_R = 0$	$\varphi_L = \dfrac{\pi}{2}$	$\varphi_C = -\dfrac{\pi}{2}$
Représentation de Fresnel	 Le courant est en phase avec la tension	 Le courant est en retard de $\pi/2$ sur la tension	 Le courant est en avance de $\pi/2$ sur la tension
Relations de phase			

3.3. Passive Dipole Grouping

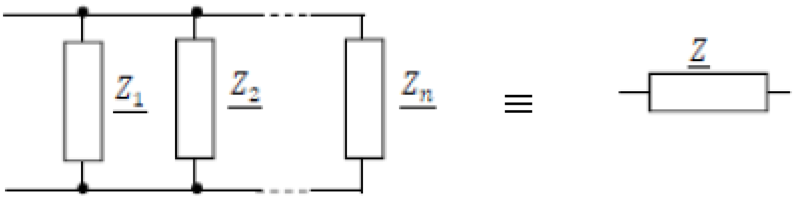
Let be a grouping of passive dipoles, with complex impedance Zi and complex admittance $\underline{Yi}=1/\underline{Zi}$.

The equivalent impedance is :

Series association

$$\underline{Z} = \sum_{i=1}^{n} \underline{Z_i}$$

Parallel association

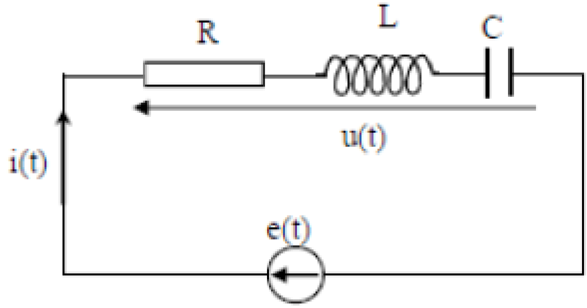


$$\underline{Y} = \sum_{i=1}^{n} \underline{Y}_i$$

What was true for the association of resistors remains applicable to the association of impedances.

3.3. Study of a serial RLC circuit

A sinusoidal generator delivering a voltage e(t), an ohmic conductor of resistor R, a perfect coil of inductance L and a capacitor of capacitance C.



With: e(t) = E√2cosωt and i(t) = I√2cosωt

$U_R = Ri(t)$, $U_L = Ldi/dt$ et $U_C = (1/C).\int idt$

Loi des mailles : U(t) = $U_R + U_L + U_C = Ri(t) + Ldi/dt + (1/C)\int idt = e$

$e = Ri(t) + Ldi/dt + (1/C)\int idt$

We replace i(t) and e(t) with their complex notation :

$\underline{E} = R\underline{I} + jL\omega\underline{I} + \frac{1}{jC\omega}\underline{I} = (R + J\left(L\omega - \frac{1}{C\omega}\right))\underline{I}$, So, we have : $\underline{U} = \underline{Z}\underline{I}$

We find the impedance: $\underline{Z} = R + j(L\omega - 1/C\omega)$, and its modulus : $|\underline{Z}| = \sqrt{R^2 + (L\omega - 1/C\omega)^2}$

And the argument: $\varphi = \arctan \frac{(L\omega - \frac{1}{jC\omega})}{R}$

Resonance in intensity

In a series RLC circuit, when the generator imposes a pulsation $\omega = \omega_0$ (the proper pulsation), the circuit enters into intensity resonance, the intensity of the current is then maximum :

$$\underline{I} = \frac{\underline{E}}{\underline{Z}} = \frac{\underline{E}}{R + j(L\omega - \frac{1}{C\omega})}$$

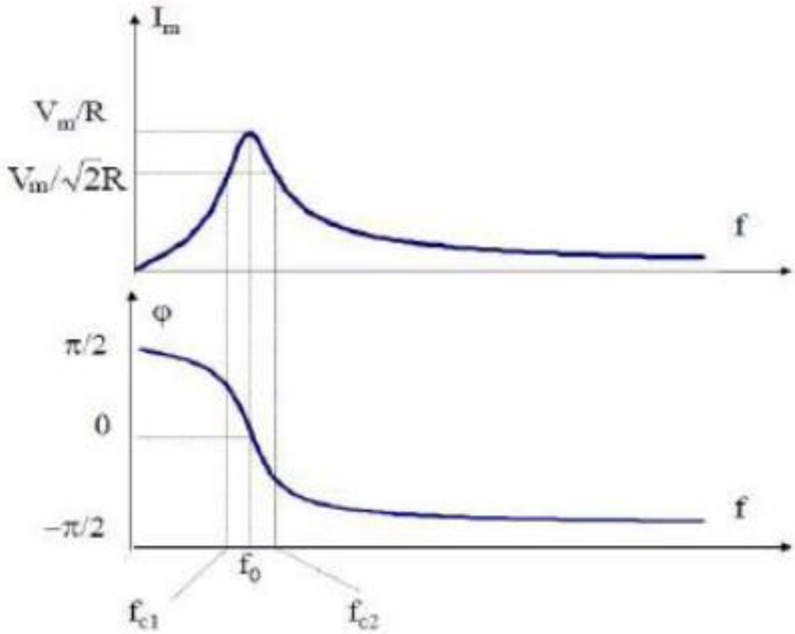
$\underline{I}$ is maximum, when the denominator is minimal : Lω – (1/Cω) = 0. So we will have :

- Proper pulse ω_0 : $LC\omega_0^2 = 1, \omega_0 = \frac{1}{\sqrt{LC}}$

- Natural frequency f_0 : $f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$

- The impedance of the circuit is minimal and real: Z = R

- The phase shift is zero: φ = 0



- Bandwidth: $\Delta\omega = \omega_2 - \omega_1$, ω_1 and ω_2 are values of ω for which $I = I_m/\sqrt{2}$

  The width of this bandwidth is: $\Delta\omega = \omega_2 - \omega_1 = \omega_0/Q$

- The resonance acuity is expressed using the quality factor of the circuit Q :

$Q = \frac{L\omega_0}{R} = \frac{1}{RC\omega_0} = \frac{1}{R}\sqrt{\frac{L}{C}}$, The greater the quality factor, the higher the resonance.

3.5. Puissance

3.5.1. Puissance instantanée

Let us say: $u(t) = U\sqrt{2}\cos(\omega t + \varphi)$ the voltage across a dipole,

$i(t) = I\sqrt{2}\cos\omega t$ the intensity of the current that passes through it.

The power received by this dipole is defined by: p(t) = u(t).i(t).

$p(t) = 2U.I\cos(\omega t)\cos(\omega t + \varphi) = UI\cos\varphi + UI\cos(2\omega t + \varphi)$

It is noted that power is a periodic function of period T/2 with respect to u(t) and i(t).

3.5.2. Average power

The average power P is defined by : $p(t) = (1/T).\int p(t)dt = UI\cos\varphi$

Power is also called active power in sinusoidal regime.

The perfect coil ($\varphi = \pi/2$) and the capacitor ($\varphi = \pi/2$) do not consume active power and therefore no energy.