Chapter 4

Partial Differential Equations

4.1 General Definitions

Let u be a function depending on n independent variables $x_1, x_2, \ldots, x_n$.

Definition 4.1 *A **partial differential equation (PDE)** is any relation involving an unknown function u of several independent variables and its partial derivatives.*

$$F(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, ..., \frac{\partial^m u}{\partial x_n^m}) = 0. \qquad (E)$$

The most common examples of linear partial differential equations are:

- *The heat equation (or diffusion equation) in one-dimensional:* $\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$.
- *The wave equation (propagation equation) in one-dimensional:* $\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$.
- *The Laplace equation in two-dimensional:* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

The solutions of the partial differential equation (E) are the functions that satisfy this equation in Ω a domain of $\mathbb{R}^n$.

To find particular solutions of a PDE from the general solution, restrictive conditions are often imposed on the set of solutions. The most common conditions are:

1. ***Initial conditions**: If u is a function of (x, t) that satisfies the PDE (E), we impose $u(x, t_0) = f_0(x)$. These are also referred to as Cauchy conditions.*

2. ***Boundary conditions***: *If u is a function of x that satisfies the PDE (E) in the region Ω, two types of constraints can be applied:*

   - ***Dirichlet conditions***, *where u is specified on the boundary $\partial\Omega$ of Ω:*

$$u|_{\partial\Omega} = g.$$

   - ***Neumann conditions***: *where the normal derivative of u, $\frac{\partial u}{\partial n} = \nabla u \cdot n$, is specified on the boundary $\partial\Omega$ of Ω:*

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g,$$

     where n is the outward normal to Ω.

Definition 4.2 *The **order** of a PDE is the order of the highest partial derivative it contains.*

Definition 4.3 *A **PDE** is said to be **linear** if it is linear in the unknown function u and its partial derivatives.*

A Partial Differential Equation is called quasi-linear if it is linear with respect to the highest-order partial derivatives.

Example 4.1 *:*

1. *The equation:*

$$\left(\frac{\partial u}{\partial x}\right)^2 - u\frac{\partial u}{\partial y} = 0$$

   is a first-order, non-linear partial differential equation.

2. *The equation:*

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} = \alpha\nabla^2 u, \quad (x,t) \in \Omega \times (0,T],$$

   is the heat equation, which is a second-order linear PDE where u represents temperature as a function of time t and position x, and α is a diffusion constant.

   - *The initial condition at the initial time $t = 0$, the temperature distribution is specified as:*

$$u(x,0) = f(x), \quad x \in \Omega.$$

- *The problem may also include boundary conditions, such as:*
  - ***Dirichlet conditions**: $u(0,t) = 0$ and $u(L,t) = 0$, representing a rod with ends kept at zero temperature.*
  - ***Neumann conditions**: $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$, representing insulated ends.*

3. *The Laplace equation:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is a second-order, linear, homogeneous PDE.

4.2 First-Order Partial Differential Equations

***Definition 4.4** A first-order partial Differential Equations is of the form:*

$$F\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right) = g(x_1, \ldots, x_n).$$

A first-order quasi-linear PDE with two variables is of the form:

$$f(x,y,u)\frac{\partial u}{\partial x} + g(x,y,u)\frac{\partial u}{\partial y} = h(x,y,u(x,y)).$$

***Example 4.2** The transport equation:*

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

models the transport of a quantity along a flow and is a first-order PDE.

4.2.1 Resolution methods of first order partial differential equations

*In the following, we will introduce two analytical methods for solving partial differential equations (PDEs): The **Method of Characteristics** and the **Separation of Variables**.*

I. Method of Characteristics

The method of characteristics is used to convert a partial differential equation (PDE) into a system of ordinary differential equations.

- *We look for the solution of a first-order PDE of the following form:*

$$f(x, y, u(x, y))\frac{\partial u}{\partial x} + g\left(x, y, u(x, y)\right)\frac{\partial u}{\partial y} = h(x, y, u(x, y)), \tag{$*$}$$

where f, g, and h are given functions defined on a subset of $\mathbb{R}^3$.

- *We set up the characteristic equations of ($*$) obtained by equating the ratios of the coefficients of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, and the derivative of u. The characteristic equations are:*

$$\frac{dx}{f(x, y, u)} = \frac{dy}{g(x, y, u)} = \frac{du}{h(x, y, u)}$$

This system is a system of ordinary differential equations (ODEs).

- *Solve the Characteristic Equations*
The solutions of the characteristic equation are defined by $\varphi(c_1, c_2) = 0$ or $c_1 = \varphi(c_2)$, c_1 and c_2 are given by Solved the following ordinary differential equations (ODEs):

1. *Solve $\frac{dx}{f(x,y,u)} = \frac{dy}{g(x,y,u)}$ to find the characteristic curves $c_1 = y(x)$ or their parametric form.*

2. *Solve $\frac{dx}{f(x,y,u)} = \frac{du}{h(x,y,u)}$ and/or $\frac{dy}{g(x,y,u)} = \frac{du}{h(x,y,u)}$ to find the relationship between x, y, and u, i.e. ($c_2 = u(x)$ or $c_2 = u(y)$).*

Example 4.3 *Solve the equation:*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u. \tag{E}$$

We have:

$$f(x, y, u) = x,$$

$$g(x, y, u) = y,$$

$$h(x, y, u) = u.$$

Thus, the characteristic system of (E) is:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}.$$

The first integrals are given by:

$$\begin{cases} \int \frac{dx}{x} = \int \frac{dy}{y} \\ \int \frac{dx}{x} = \int \frac{du}{u} \end{cases} \Rightarrow \begin{cases} \ln x = \ln y + c_1 \\ \ln x = \ln u + c_2 \end{cases} \Rightarrow \begin{cases} ln\left(\frac{y}{x}\right) = c_1 \\ \ln\left(\frac{x}{u}\right) = c_2 \end{cases} \Rightarrow \begin{cases} \frac{y}{x} = c_1 \\ \frac{x}{u} = c_2 \end{cases}$$

The general solution of (*) is:

$$\varphi\left(c_1, c_2\right) = \varphi\left(\frac{y}{x}, \frac{u}{x}\right) = 0$$

Or $\frac{u}{x} = \varphi\left(\frac{y}{x}\right) \Rightarrow u = x\varphi\left(\frac{y}{x}\right)$

Example 4.4 *Solve the equation:*

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0.$$

We have:

$$f(x, y, u) = y,$$

$$g(x, y, u) = -x,$$

$$h(x, y, u) = 0.$$

The characteristic equations are:

$$\frac{dx}{y} = \frac{dy}{-x} = 0.$$

Solving,

$$\int \frac{dx}{y} = \int \frac{dy}{-x} \implies \int -xdx = \int ydy$$

We find:

$$x^2 + y^2 = c_1, \quad u = c_2.$$

Thus, the general solution is:

$$\varphi(c_1, c_2) = \varphi(x^2 + y^2, u),$$

where φ is an arbitrary function.

So the solution is:

$$u(x, y) = \varphi(x^2 + y^2),$$

Remark 4.1 *φ is an arbitrary function. Arbitrary functions are determined by initial conditions.*

Example 4.5 *Solve the following equation problem:*

$$(E)\begin{cases} \frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = u, \\ u(0,y) = \sin(y). \end{cases}$$

The system has the following characteristics:

$$\frac{dx}{1} = \frac{dy}{x} = \frac{du}{y}.$$

This implies:

$$\Rightarrow \begin{cases} x\,dx = dy \\ dx = \frac{du}{u} \end{cases}$$

From these equations, we have:

$$\begin{cases} c_1 = y - \frac{1}{2}x^2 \\ c_2 = ue^{-x} \end{cases}$$

Thus, the general solution is:

$$\varphi(c_1, c_2) = \varphi\left(y - \frac{1}{2}x^2, ue^{-x}\right).$$

Or, we can express it as:

$$u(x,y) = \varphi\left(y - \frac{1}{2}x^2\right)e^x.$$

Given that:

$$u(0,y) = \sin(y),$$

we have:

$$\Rightarrow u(0,y) = e^0\,\varphi(y).$$

Thus,

$$\varphi(y) = \sin(y)$$

.

Finally, the particular solution of (E) is:

$$\boxed{u(x,y) = e^x \sin\left(y - \frac{1}{2}x^2\right).}$$

4.3 Separation of Variables Method

Let the following partial differential equation be given:

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = 0 \quad \cdots (E)$$

where A and B are functions of a single variable.

- *Assume:*

$$u(x,y) = X(x)Y(y)$$

Thus:

$$\frac{\partial u}{\partial x} = X'(x)Y(y), \quad \frac{\partial u}{\partial y} = X(x)Y'(y).$$

- *Substituting into (E):*

$$AX'(x)Y(y) + BX(x)Y'(y) = 0.$$

Therefore, we obtain the following system of ODEs:

$$\frac{AX'(x)}{X(x)} = -\frac{BY'(y)}{Y(y)} = K,$$

where K is a constant.

- *Integrate both ODEs and substitute $X(x)$ and $Y(y)$ into $u(x,y)$.*

Example 4.6 *Solve the following problem:*

$$y\frac{\partial u}{\partial x} + x^2\frac{\partial u}{\partial y} = 0 \quad \cdots (E).$$

1. *Assume $u(x,y) = X(x)Y(y)$. Thus:*

$$\frac{\partial u}{\partial x} = X'(x)Y(y), \quad \frac{\partial u}{\partial y} = X(x)Y'(y).$$

2. *Substitute into (E):*

$$yX'(x)Y(y) + x^2X(x)Y'(y) = 0.$$

Therefore, we obtain the system of ODEs:

$$\frac{yX'(x)}{X(x)} = -\frac{x^2Y'(y)}{Y(y)} = k \quad \textit{(constant)}.$$

$$\frac{-1}{x^2}\frac{X'(x)}{X(x)} = \frac{1}{y}\frac{Y'(y)}{Y(y)} = k \quad \textit{(constant)}.$$

3. *Solve the first equation:*

$$\frac{-1}{x^2}\frac{X'(x)}{X(x)} = k \quad \Rightarrow \ln|X| = \frac{-k}{3}x^3 + c_1 \quad \Rightarrow \quad X(x) = C_1e^{-\frac{k}{3}x^3}.$$

Solve the second equation:

$$\frac{1}{y}\frac{Y'(y)}{Y(y)} = k \quad \Rightarrow \ln|Y| = \frac{k}{2}y^2 + c_2 \quad Y(y) = C_2e^{\frac{k}{2}y^2}.$$

4. *Substitute $X(x)$ and $Y(y)$ into $u(x,y)$:*

$$u(x,y) = C_1C_2e^{\frac{k}{2}y^2-\frac{k}{3}x^3} = Ce^{\frac{k}{2}y^2-\frac{k}{3}x^3}$$

Example 4.7 *Solve the following problem:*

$$\begin{cases} \frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} = 0, \\ u(x,0) = e^{-x^2}. \end{cases}$$

1. *Assume $u(x,y) = X(x)Y(y)$. Thus:*

$$\frac{\partial u}{\partial x} = X'(x)Y(y), \quad \frac{\partial u}{\partial y} = X(x)Y'(y).$$

2. *Substitute into (E):*

$$X'(x)Y(y) + 2xy^2X(x)Y'(y) = 0.$$

Therefore, we obtain the system of ODEs:

$$\frac{X'(x)}{X(x)} = 2kx \quad \textit{and} \quad \frac{Y'(y)}{Y(y)} = \frac{k}{y^2}.$$

3. *Solve the equations*

$$\frac{X'(x)}{X(x)} = 2kx \quad \Rightarrow \ln|X| = kx^2 + c_1 \quad \Rightarrow \quad X(x) = C_1 e^{kx^2}.$$

$$\frac{Y'(y)}{Y(y)} = \frac{-k}{y^2} \quad \Rightarrow \ln|Y| = \frac{k}{y} + c_2 \quad \Rightarrow \; Y(y) = C_2 e^{\frac{k}{y}}$$

4. *Substitute $X(x)$ and $Y(y)$ into $u(x, y)$:*

$$u(x, y) = C_1 C_2 e^{k(x^2 + \frac{1}{y})} = Ce^{k(x^2 + \frac{1}{y})}$$

with the condition

$$u(x, 0) = e^{-x^2}.$$

so we have

$$u(x, 0) = Ce^{kx^2} = e^{-x^2}, c = 1, k = -1.$$

So the particulaire solution is

$$u(x, y) = e^{-(x^2 + \frac{1}{y})}$$

4.4 Second-Order Partial Differential Equations

Let u be a function of two variables x and y. A second-order partial differential equation is defined as a relation of the form:

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0.$$

4.4.1 Classification of second-order PDEs:

Consider a second-order PDE of the form:

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = g(x,y).$$

We can write:

$$d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u - g(x,y) = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

The equation becomes:

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

Let $\Delta = b^2 - 4ac$, then we distinguish three cases:

1. *If $\Delta > 0$, the PDE is said to be of* ***Hyperbolic*** *type.*
2. *If $\Delta = 0$, the PDE is said to be of* ***Parabolic*** *type.*
3. *If $\Delta < 0$, the PDE is said to be of* ***Elliptic*** *type.*

Example 4.8 ,

1. $\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} = 0, \quad \Delta = 25 > 0,$ *This PDE is of* ***hyperbolic*** *type.*
2. $\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial u}{\partial y} = -e^{-x}, \quad \Delta = 0,$ *This PDE is of* ***parabolic*** *type.*
3. $y^2\frac{\partial^2 u}{\partial x^2} + x^2\frac{\partial^2 u}{\partial y^2} + 3u = 0, \quad \Delta = -4x^2y^2,$ *Thus, the PDE is:*
   - ***Elliptic*** *if $x \neq 0$ and $y \neq 0$,*
   - ***Parabolic*** *if $x = 0$ or $y = 0$.*

key equations of physics

Laplace's Equation

Laplace's equation is a second-order partial differential equation of the form:

$$\frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

For $c = \pm 1$, this equation is classified as ***elliptic****.*

Poisson's Equation

Poisson's equation extends Laplace's equation and has the form:

$$\nabla^2 u = \delta(x, y),$$

where $\nabla^2 u$ is the Laplacian operator, and $\delta(x, y)$ represents a source term, is classified as an ***elliptic equation****.*

Heat (or Diffusion) Equation

The heat equation, also known as the diffusion equation, is given by:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a} \frac{\partial u}{\partial t}.$$

This is classified as a ***parabolic equation****.*

Wave Equation

It is given form as:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2},$$

This is classified as a ***hyperbolic equation****.*

4.4.2 Resolution Methods for Second-Order PDEs

Characteristic Method

Step 1: *Solve the characteristic equations:*

$$A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0,$$

where A, B, and C are the coefficients of the PDE.

Cases:

- *If* $\Delta > 0$*:* $\frac{dy}{dx} = \frac{B \pm \sqrt{\Delta}}{2A}$.
- *If* $\Delta = 0$*:* $\frac{dy}{dx} = \frac{B}{2A}$.
- *If* $\Delta < 0$*:* $\frac{dy}{dx} = \frac{B \pm i\sqrt{\Delta}}{2A}$.

Integrate the two differential equations, and let:

$$c_1 = \varphi(x, y) \text{ and } c_2 = \phi(x, y).$$

Step 2: *Find the canonical form of the PDE.*

Make the following change:

$$\begin{cases} s = c_1 = \varphi(x \cdot y), \\ t = c_2 = \phi(n \cdot y). \end{cases}$$

Rewrite the PDE in terms of the new variables s and t.

Remark 4.2 *The resulting equation is called the canonical form of the PDE.*

Step 3: *Solve the new equation and rewrite the solution in terms of x and y.*

Example 4.9 *Solve the following PDE:*

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} = 0 \quad \cdots (E_1).$$

1. *Solve the characteristic equations:*

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 4\frac{\partial u}{\partial x} = 0 \quad \Rightarrow \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x} - 4\right) = 0$$

$$\Rightarrow \begin{cases} \frac{dy}{dx} = 0 \\ \frac{dy}{dx} = 4 \end{cases} \Rightarrow \begin{cases} y = c_1 \\ y = 4x + c_2 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = y \\ c_2 = y - 4x \end{cases}$$

2. *The canonical form of (E_1): Let*

$$\begin{cases} s = y \\ t = y - 4x. \end{cases}$$

Then,

$$\frac{\partial u}{\partial x} = \frac{\partial u \partial s}{\partial s \partial x} + \frac{\partial u \partial t}{\partial t \partial x} = -4\frac{\partial u}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -4\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = 16\frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -4\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) = -4\frac{\partial^2 u}{\partial s \partial t} - 4\frac{\partial^2 u}{\partial t^2}.$$

Thus, we have:

$$16\frac{\partial^2 u}{\partial t^2} - 16\frac{\partial^2 u}{\partial s \partial t} - 16\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial u}{\partial t}$$

$$\Rightarrow -16\frac{\partial^2 u}{\partial s \partial t} - 4\frac{\partial u}{\partial t} = 0 \Rightarrow 4\frac{\partial^2 u}{\partial s \partial t} + \frac{\partial u}{\partial t} = 0$$

The canonical form of (E_1).

3. *Solution of canonical equation*

$$4\frac{\partial^2 u}{\partial s \partial t} + \frac{\partial u}{\partial t} = 0 \quad \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial s} + u\right) = 0$$

$$\Rightarrow 4\frac{\partial u}{\partial s} + u = \int 0\, dt + f(s) = f(s),$$

$$\Rightarrow 4\frac{\partial u}{\partial s} + u = f(s) \quad \text{(a first-order differential equation).}$$

Solution of first-order differential equation

$$4\frac{\partial u}{\partial s} + u = f(s) \quad \cdots (*)$$

First: Solving the Homogeneous Equation

$$4\frac{\partial u}{\partial s} + u = 0$$

The solution is given by

$$u = ke^{-\frac{1}{4}s}, \quad k \in \mathbb{R}.$$

Second: The particular solution of (∗) by the method of separation of variables

We write

$$u_p = k(s)e^{-\frac{1}{4}s},$$

We get the particular solution:

$$u_p = \frac{1}{4}e^{-\frac{1}{4}s}\int f(s)e^{-\frac{1}{4}s}\,ds.$$

General Solution:

$$u(s,t) = ke^{\frac{1}{4}s} + \frac{1}{4}e^{-\frac{1}{4}s}\int f(s)e^{-\frac{1}{4}s}\,ds.$$

Example 4.10 *Solve the equation:*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

1. *The characteristic equation:*

$$A\left(\frac{dg}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$$

$$\Leftrightarrow \left(\frac{d^2y}{dx^2}\right)^2 - 1 = 0 \Rightarrow \frac{dy}{dx} = \pm 1$$

$$\Rightarrow \begin{cases} \frac{dy}{dx} = 1 \\ \frac{dy}{dx} = -1 \end{cases} \Leftrightarrow \begin{cases} y = x + c \\ y = -x + c \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = y - x \\ c_2 = y + x. \end{cases}$$

2. *The canonical form of (E):*

$$t = y + x, s = y - x$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial x}\frac{\partial u}{\partial s} + \frac{\partial}{\partial x}\frac{\partial u}{\partial t},$$

$$= -\left(\frac{\partial}{\partial s}\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\frac{\partial U}{\partial s}\cdot\frac{\partial t}{\partial x}\right)$$

$$+\frac{\partial}{\partial s}\frac{\partial U}{\partial t}\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\frac{\partial U}{\partial t}\frac{\partial t}{\partial x},$$

$$= \frac{\partial^2 U}{\partial s^2} - \frac{\partial^2 U}{\partial t\partial s} - \frac{\partial^2 U}{\partial s\partial t} + \frac{\partial^2 U}{\partial t^2},$$

$$= \frac{\partial^2 U}{\partial s^2} - 2\frac{\partial^2 U}{\partial s\partial t} + \frac{\partial^2 U}{\partial t^2}.$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial U}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial U}{\partial s} + \frac{\partial U}{\partial t},$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial s} + \frac{\partial U}{\partial t}\right),$$

$$= \frac{\partial}{\partial y}\frac{\partial U}{\partial s} + \frac{\partial}{\partial y}\frac{\partial U}{\partial t},$$

$$= \frac{\partial}{\partial s}\frac{\partial U}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial}{\partial t}\frac{\partial U}{\partial s}\cdot\frac{\partial t}{\partial y},$$

$$+\frac{\partial}{\partial s}\frac{\partial U}{\partial t}\cdot\frac{\partial s}{\partial y} + \frac{\partial}{\partial t}\frac{\partial U}{\partial t}\cdot\frac{\partial t}{\partial y},$$

$$= \frac{\partial^2 U}{\partial s^2} + 2\frac{\partial^2 U}{\partial s\partial t} + \frac{\partial^2 U}{\partial t^2}.$$

Thus, the canonical form of (E) is:

$$\frac{\partial^2 U}{\partial s^2} - \frac{\partial^2 U}{\partial s\partial t} + \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial s^2} - 2\frac{\partial^2 U}{\partial s^2} - \frac{\partial^2 U}{\partial t^2} = 0,$$

$$\Rightarrow -4\frac{\partial^2 U}{\partial s\partial t} = 0 \Rightarrow \frac{\partial^2 U}{\partial s\partial t} = 0.$$

3. *The solution to the equation:*

$$\frac{\partial^2 U}{\partial s\partial t} = 0 \quad \textit{is} \quad U = ?$$

$$\frac{\partial^2 U}{\partial s\partial t} = 0 \Rightarrow \frac{\partial}{\partial s}\left(\frac{\partial U}{\partial t}\right) = 0,$$

$$\Rightarrow \frac{\partial U}{\partial t} = f(t), \quad \textit{where} \quad A\cos\theta - \cos a',$$

$$U(s,t) = \int f(t)\,dt + g(s).$$

$$\Rightarrow U(s,t) = F(t) + g(s)$$

Here, F and g are arbitrary functions. The general solution of (E) is:

$$U(x,y) = F(y+x) + g(y+x).$$