4^{th} Tutorial Series
"Partial Differential Equations "

2^{nd} Year Engineering (S3)

Exercise 1: Determine, for each PDE, whether it is linear, homogeneous, and its order.

1. $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$,
2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$,
3. $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(t, x)$,
4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$,
5. $\left(\frac{\partial u}{\partial t}\right)^2 + u \frac{\partial u}{\partial x} = 0$,
6. $\frac{\partial^2 u}{\partial y^2} + a \frac{\partial u}{\partial x} + b u^3 = 0$.

a, b, and k are constants, and f is a given function.

Exercise 2:

1. Solve the following first-order PDEs using the method of characteristics:

   1. $y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = 0$,
   2. $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2xy$,
   3. $x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = u^2$,
   4. $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0$.

2. Solve the problem:

$$\begin{cases} \frac{\partial u}{\partial t} + e^x \frac{\partial u}{\partial x} = 0, \\ u(x, 0) = x. \end{cases}$$

Exercise 3:

Solve the following first-order PDEs using the separation of variables method:

1. $\begin{cases} \frac{\partial u}{\partial x} - 2xy^2 \frac{\partial u}{\partial y} = 0, \\ u(0, y) = 2e^{\frac{2}{y}}. \end{cases}$
2. $\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x + y)u, \\ u(x, 0) = e^{x^2 + x}. \end{cases}$

Exercise 4

1. Determine the type and the canonical form of the following PDEs:

$$(E_1) : \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} = 0,$$
$$(E_2) : \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$
$$(E_3) : \frac{\partial^2 u}{\partial x^2} - x \frac{\partial^2 u}{\partial y^2} = 0,$$
$$(E_4) : \frac{\partial^2 u}{\partial x^2} - 6 \frac{\partial^2 u}{\partial x \partial y} + 9 \frac{\partial^2 u}{\partial y^2} = xy^2.$$

2. Solve the following PDEs using the method of characteristics:

$$(E_1) : \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} = 0,$$
$$(E_2) : \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

3. Solve the following problem:

$$(P) : \begin{cases} \frac{\partial^2 u}{\partial x^2} - 6 \frac{\partial^2 u}{\partial x \partial y} + 9 \frac{\partial^2 u}{\partial y^2} = xy^2, \\ u(0, y) = 0, \\ u(x, 0) = 0. \end{cases}$$

Exercise 5

Using the method of separation of variables, solve the following problem:

$$(P):\begin{cases}\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}, & x\in(0,l),\ t>0,\\ u(0,t)=u(l,t)=0. & \end{cases}$$